# Game Theory

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Home Page of Game Theory:

http://www.luigicurini.com/game-theory-for-social-scientists1.html

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# Game Theory

1: The theory of choice: the individual

# Game Theory and Social Science

- Game Theory (GT) is the application of **formal deductive reasoning** to **social interactions**, that is, GT analyses in a deductive way situations where people **interact** in an **interdependent** way
- **Interdependency** means that people are **conscious** that others' behavior influences their own conditions, and **react** to this knowledge

# Strategic approach to social interactions

- This can be simply rephrased saying that GT assumes that people act **strategically**, that is…

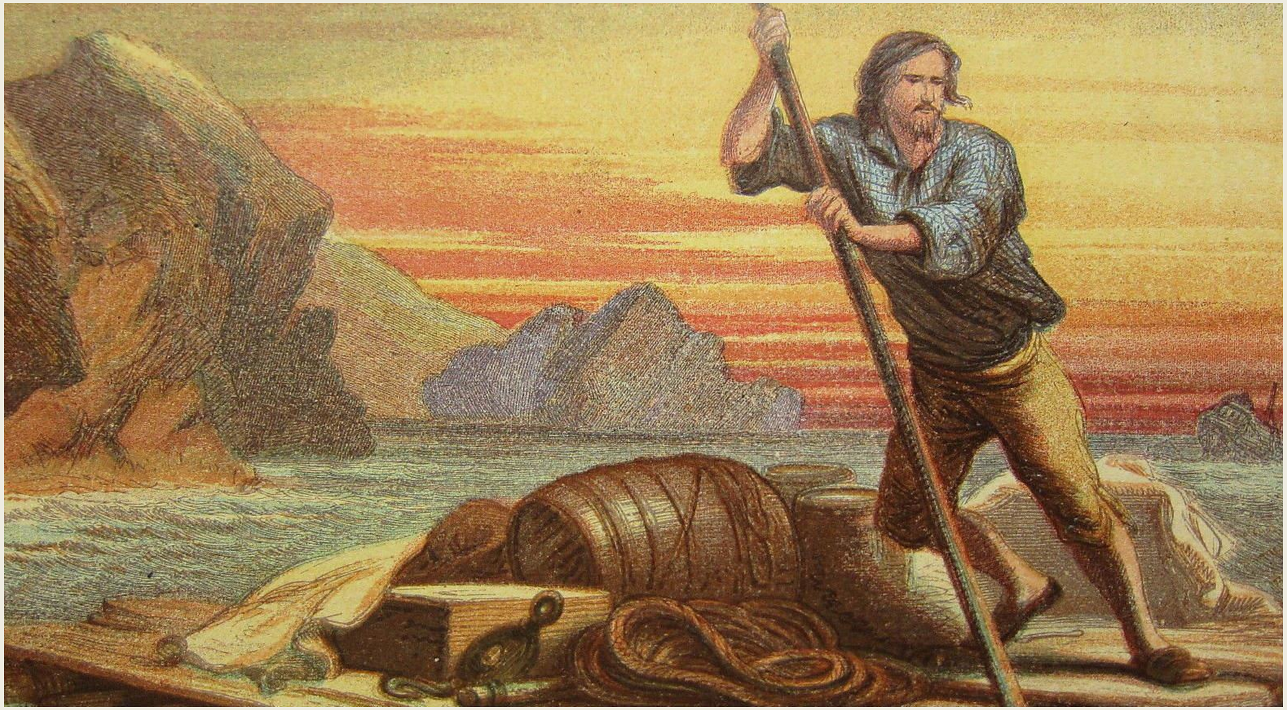
**…Games are models of strategic behavior in social interaction**

# Strategic approach to social interactions

- We will learn how to arrive at formal conclusions about how actors **play these games**, given their **interests**, their **information**, and the **constraints** with which they are bound to act in an **environment** governed by rules and institutions

# The preamble to GT

- Before dealing with people interacting strategically we need **however** to know how **isolated individuals** behave



# The preamble to GT

- The **theory of individual choice** is the first step into the game theoretical analysis
- Three concepts must be clearly defined in this respect:

1. choice (and choice set)
2. utility (and utility function)
3. rationality

# Choice and choice set

- Any kind of meaningful **action** is a **choice** (of behavior)
- A **choice set** is the collection of **exhaustive** and **mutually exclusive** alternatives among which the choice takes place
- **Exhaustive** means that the choice set includes all possible alternative
- **Mutually exclusive** means that one and only one alternative is bound to happen

# Utility and the choice set

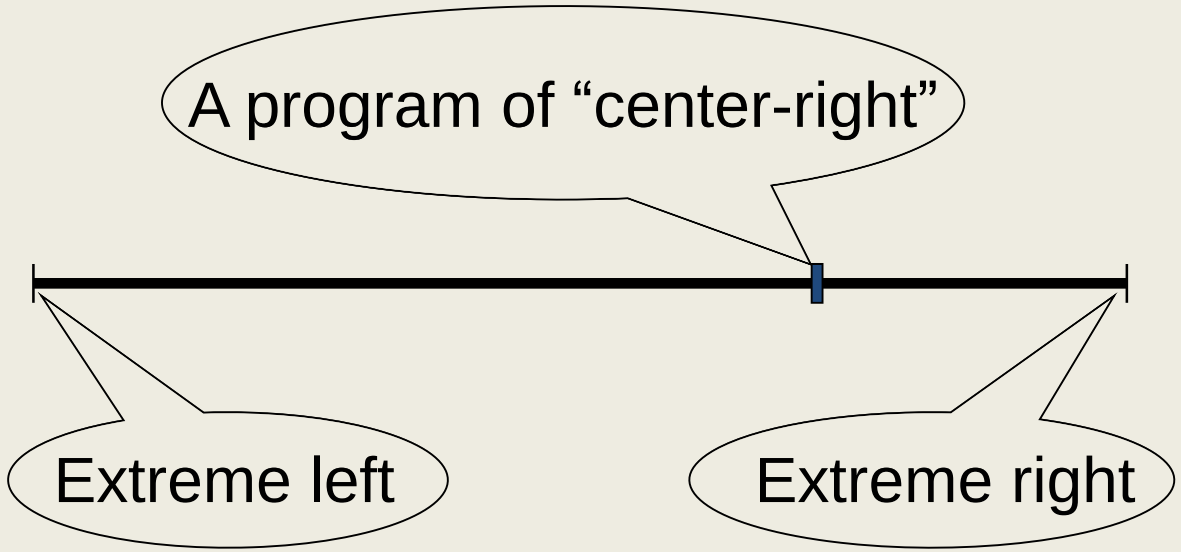
- Choice sets are formalized in such a way that **any alternative** can be distinguished by the **values** of one or more **variables**

# Example: choice of the city to visit

| cities | | | | | |
|---|---|---|---|---|---|
| MILAN | ROME | LONDON | NEW YORK | TOKYO | … |

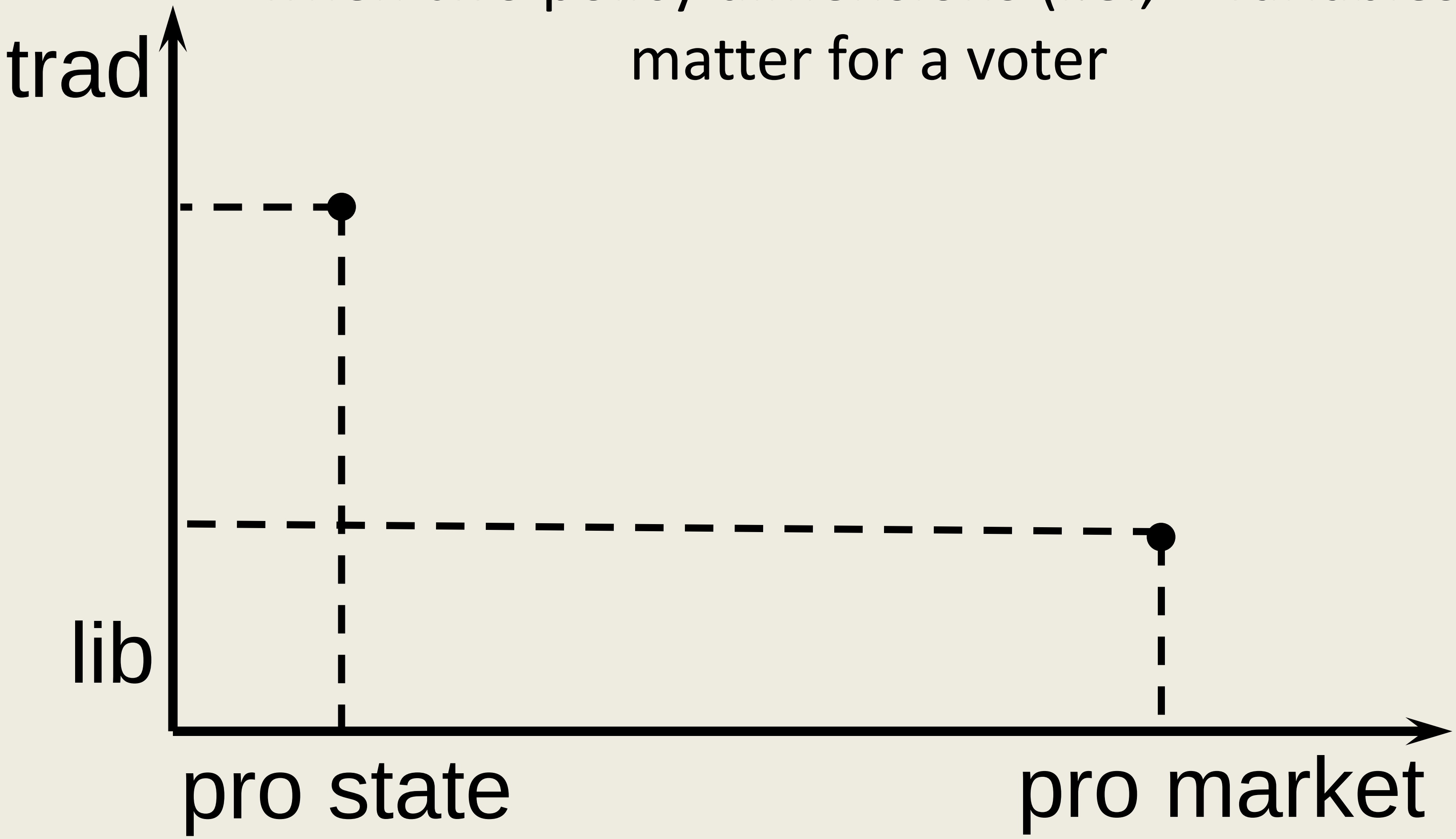
- The **variable** is “city” and the **values** of the variable are the “names of the cities”

# Choice of a political ideology




- The variable is **ideology** and the **values** are all possible ideological positions on the left-to-right space

Of course more than one variable can affect a choice. Ex: choice of an electoral program when two policy dimensions (i.e., 2 variables) matter for a voter



# Utility: formal definition

- The **formal theory of choice** is based on the concept of **utility**
- **Utility** is
  - the **value** the individual chooser associates to what she gets from the choice
  - an index of that value can be expressed by either a **number** **(i.e., 10>6>5)** or a **parameter** **(i.e., a>b>c)**
- Utility is therefore a **property** of the alternatives in the **choice set**

# Utility: formal definition

- The higher the utility, the **more preferred** is the alternative
- The individual chooses the alternative that **maximizes the utility**

# Utility: formal definition

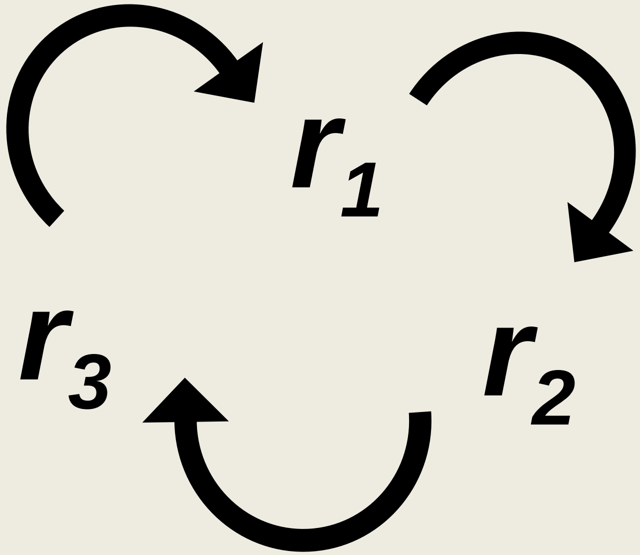
- Does the fact that individual choose the alternative that **maximizes the utility** implies that actors are assumed to be egoistic?
- **no, No, NO!**
- They can maximize **any** arguments in their utility function (including altruism, masochism, etc.)!!!

# Rationality

And **rationality**? Two axioms must be satisfied in any given choice so that rationality can be satisfied:

- ✓ **Completeness**: a rational actor is able to **compare** any couple of alternatives and to choose between the two
- ✓ **Transitivity**: if a rational actor (weakly) prefers alternative A to alternative B and (weakly) alternative B to alternative C **then** she (weakly) prefers alternative A to alternative C

# Transitivity allows to **avoid a cycle**…

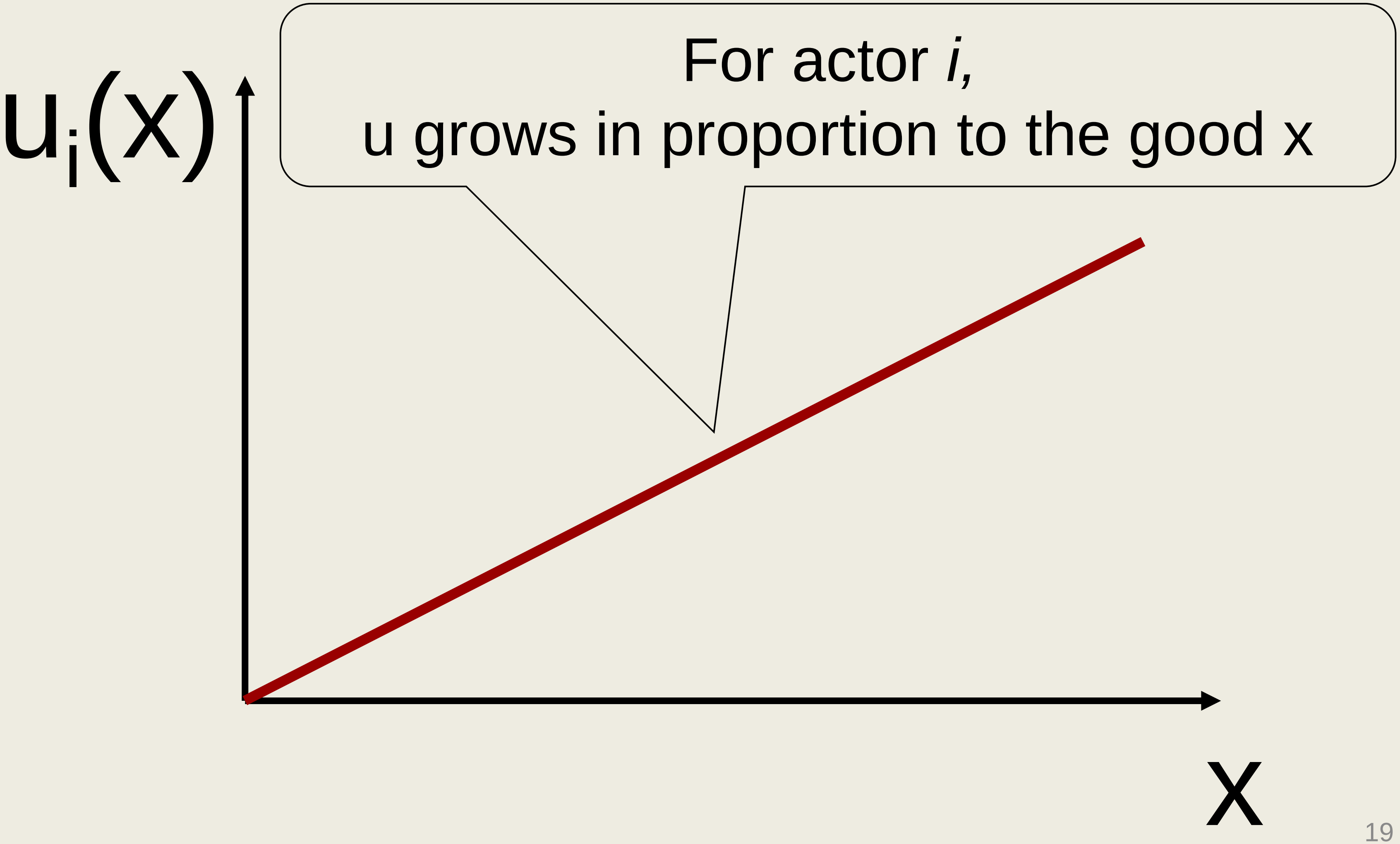



…and therefore to choose an alternative to maximize one's utility! (at least at the individual level…not at the aggregate level: the **Impossibility theorem** of *Arrow*!)

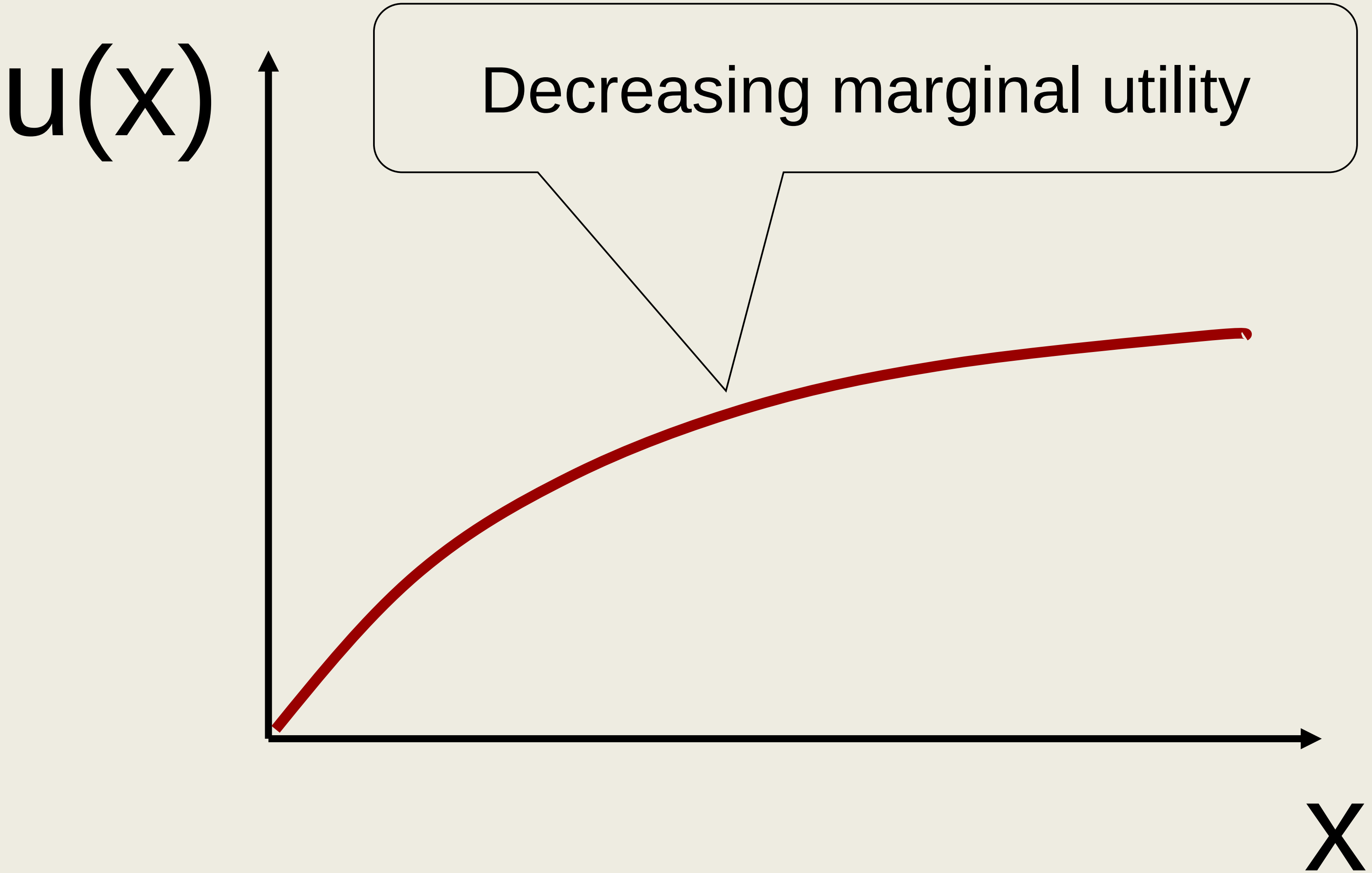
# Utility function

- choice → result (r)
- result → utility (u)
- **utility function** is the correspondence between results and utilities
- **u = u(r)**

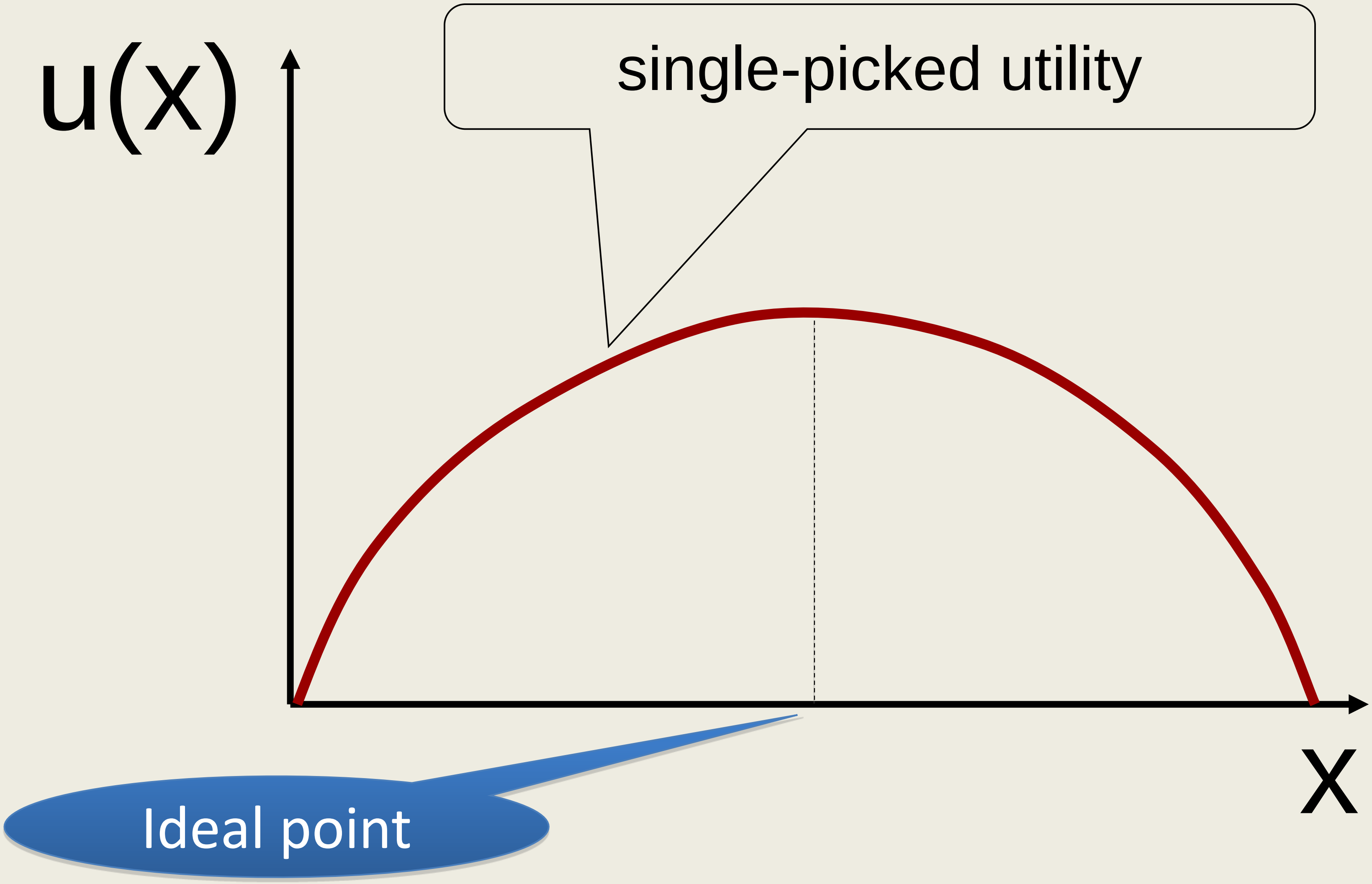
# Utility as a linear function



# Utility as a non-linear function



# Utility as a non-linear function



# A variety of utility functions!

The below functions are all plausible utility functions of the variable *x*, each of them with its specific functional form…

$u_1(x) = 4x + 2$

$u_2(x) = 12\sqrt{x}$

$u_3(x) = 30 + 6x - x^2$

# Uncertain choice

- Sometimes “**uncontrollable events**” may interfere with our choice to produce the result
- In such cases, what we get from choice depends on a **combination** between our action and **Nature**’s (i.e., fate/destiny/exogenous factors etc.) intervention
- For that reason **choice is generally uncertain**

A *theory of uncertain choice* is therefore needed!!!

# Example: How to go to work

| | sunshine | rain |
|---|---|---|
| bicycle | **happy** | **unhappy** |
| car | **unhappy** | **happy** |

Sunshine and Rain are called
«**States of the world**»

# States of the world

| | | states of the world | | | |
|---|---|---|---|---|---|
| | | $\boldsymbol{m_1}$ | $\boldsymbol{m_2}$ | **...** | $\boldsymbol{m_k}$ |
| alternatives | $\boldsymbol{s_1}$ | $r_{11}$ | $r_{12}$ | ... | $r_{1k}$ |
| | $\boldsymbol{s_2}$ | $r_{21}$ | $r_{22}$ | ... | $r_{2k}$ |
| | **...** | ... | ... | ... | ... |
| | $\boldsymbol{s_j}$ | $r_{j1}$ | $r_{j2}$ | ... | $r_{jk}$ |

# Example: how to invest savings

| | political stability | political instability |
|---|---|---|
| shares | ***1.5*** | ***0.7*** |
| real estate | ***0.9*** | ***1.2*** |

Let's suppose that:

*Probability of political stability* = 0.35

*Probability of political instability* = 0.65

# Lotteries

- ***shares lottery***:
  - *result 1.5 with probability =0.35*
  - *result 0.7 with probability=0.65*
- ***real estate lottery*:**
  - *result 0.9 with probability=0.35*
  - *result 1.2 with probability=0.65*

# Expected Results (money)

$ER(shares) = 1.5 \cdot 0.35 + 0.7 \cdot 0.65 = 0.98$

$ER(real\ estate) = 0.9 \cdot 0.35 + 1.1 \cdot 0.65 = \mathbf{1.09}$

- So should you choose Real Estate? Not necessarily…it depends on **your utility function**!

# Results, expected results
# Utilities, expected utilities

| | political stability | political instability |
|---|---|---|
| shares | **1.5** | **0.7** |
| real estate | **0.9** | **1.2** |

| | political stability | political instability |
|---|---|---|
| shares | **u(1.5)** | **u(0.7)** |
| real estate | **u(0.9)** | **u(1.2)** |

# Example: u(money) = money

| | political stability | political instability |
|---|---|---|
| shares | u(1.5) | u(0.7) |
| real estate | u(0.9) | u(1.2) |

prob(polstab)=0.35
prob(polinstab)=0.65

$$EU_{shares} = u(1.5) \times 0.35 + u(0.7) \times 0.65 = 0.98$$
$$EU_{real\ est} = u(0.9) \times 0.35 + u(1.2) \times 0.65 = \mathbf{1.09}$$

You should choose REAL ESTATE iff:

**u(money) = money**

The previous utility function (linear)
refers to a **risk neutral actor** (more on this later on)

# Example: calculus

| | political stability | political instability |
|---|---|---|
| shares | u(1.5) | u(0.7) |
| real estate | u(0.9) | u(1.2) |

prob(polstab)=0.35
prob(polinstab)=0.65

However, let us suppose: **$u(\text{money}) = 2.2 - \text{money}^2$;**
then we have:

$$EU_{shares} = [2.2-(1.5)^2] \times 0.35 + [2.2-(0.7)^2] \times 0.65 = \mathbf{1.09}$$
$$EU_{real\ est} = [2.2-(0.9)^2] \times 0.35 + [2.2-(1.2)^2] \times 0.65 = 0.98$$

The previous utility function (concave: $\cap$)
refers to a **risk averse actor** (more on this later on)

# Exercise

| | | States of the world | | |
|---|---|---|---|---|
| | | $m_1$ | $m_2$ | $m_3$ |
| Available alternatives | $s_1$ | 1 | 2 | 4 |
| | $s_2$ | 3 | 5 | 6 |

Data:

$p(m_1) = 0.3$ ; $p(m_2) = 0.5$

$u(x) = 40 - x^2$

# Risk preferences: an example

- The leader of a small party has to choose between two kinds **of electoral campaign**
- $s_1$ (traditional campaign) => N votes **with certainty**
- $s_2$ (innovative campaign) => **lottery**: (50%, N/2 votes; 50%, 3N/2)

# Risk preferences: definitions

It is *impossible* to anticipate the choice of the leader without making some assumption on her **utility function** (i.e., how to map votes into utility)

In this example, the **risk preferences** of the leader matters a lot in defining her utility function!

Does she prefer a **certain result** [50 euros] or a lottery of results [0 euro with p=1/2, 100 euro with p=1/2] that gives her an **expected result** of exactly 50 euros?

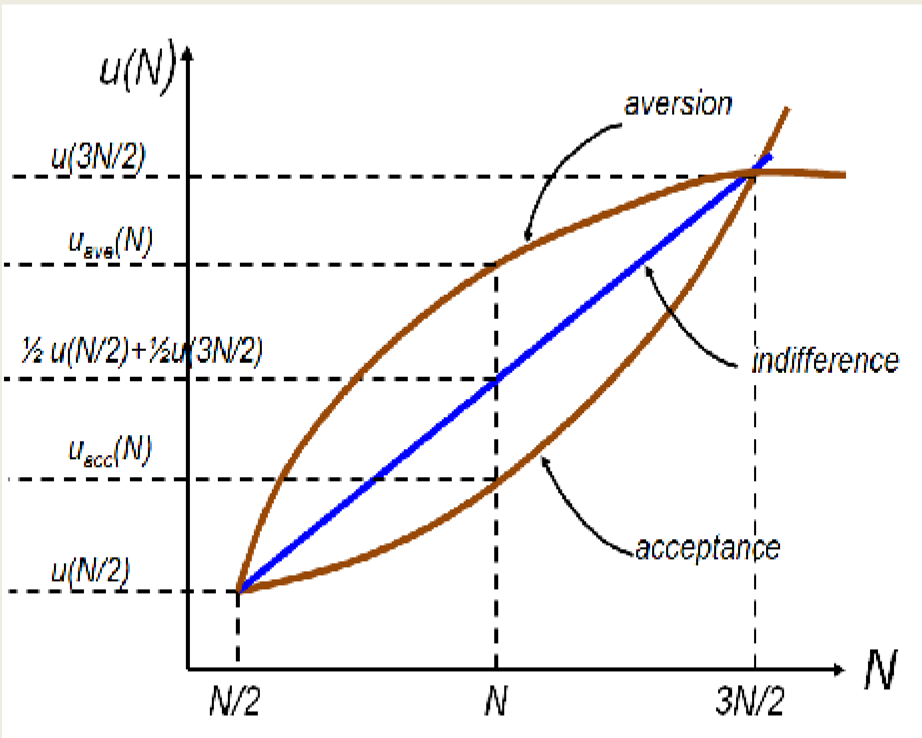
# Risk preferences: definitions

The party leader is called to compare the **certain utility** vs. the **expected utility** of getting some amount of votes:

- ✓ **risk neutral** if $u(N) = EU(\frac{1}{2}\cdot(N/2) + \frac{1}{2}\cdot(3N/2))$
- ✓ **risk averse** if $u(N) > EU(\frac{1}{2}\cdot(N/2) + \frac{1}{2}\cdot(3N/2))$
- ✓ **risk acceptant** $u(N) < EU(\frac{1}{2}\cdot(N/2) + \frac{1}{2}\cdot(3N/2))$

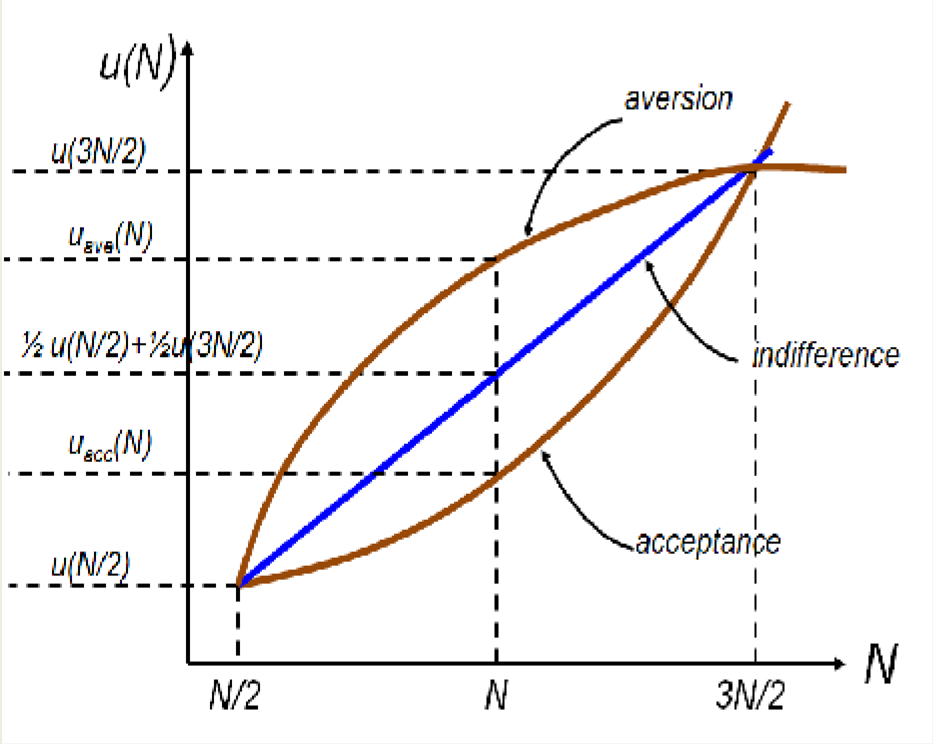
Each of this possibility produces a different utility function with a different shape!

# Risk preferences: figure




- Votes in abscissa and utility (& expected utility) for votes in ordinate
- The blue line grows proportionally with votes ➔ it represents the **expected utility** of the lottery for a **risk neutral** individual
- Here the utility for N certain votes is exactly the same as the expected utility of getting a lottery of N votes

# Risk preferences: figure




- The upper brown curve (concave: ∩) is a **risk averse** utility curve (such as y=x-x^2) – here the utility for N certain votes is higher than the expected utility of getting a lottery of N votes
- The lower brown curve (convex: ∪) is a **risk acceptant** utility curve (such as y=x+x^2) – here the utility for N certain votes is lower than the expected utility of getting a lottery of N votes

# Summary of choice under uncertainty (1)

- **choices**
- **states of the world**
- **results**
- These three elements are gathered in the matrix

| | $m_1$ | $m_2$ | … | $m_k$ | … |
|---|---|---|---|---|---|
| $S_1$ | $r_{11}$ | $r_{12}$ | … | $r_{1k}$ | … |
| $S_2$ | $r_{21}$ | $r_{22}$ | … | $r_{2k}$ | … |
| … | … | … | … | … | … |
| $S_j$ | $r_{j1}$ | $r_{j2}$ | … | $r_{jk}$ | … |
| … | … | … | … | … | … |

# Summary of choice under uncertainty (2)

- The uncertainty on the state of the world leads to **lotteries of results**
- $L = (r_1, p_1; r_2, p_2, \ldots; r_k, p_k; \ldots)$ where $p_1 + p_2 + \ldots + p_k + \ldots = 1$
- Preference between two lotteries is determined by the values of their **expected utility**
- $EU(L) = u(r_1) \cdot p_1 + u(r_2) \cdot p_2 + \ldots + u(r_k) \cdot p_k + \ldots$
- **Risk preference** is given by the shape of utility function: **concave** ∩ curves are risk averse, **convex** curve are risk acceptant, **linear** curves are risk neutral

# Parameters better than numbers!

- Employing **parameters** rather than **numbers** in your model allows you to derive more interesting (and flexible) insights
- This is true not only with games, but also with non-strategic models as the ones we are discussing in this lesson

# The voter dilemma: to vote or not to vote?

| | | States of the world | |
|---|---|---|---|
| | | Your party wins the election (p) | Your party does not win the election(1-p) |
| Available alternatives | To vote | | |
| | Not to vote | | |

**Our assumptions:**

1) Your choice does not affect the value of (p) per-se
2) The voter is risk neutral (an assumption that we will make throughout the course simply for convenience)

# The voter dilemma: to vote or not to vote?

| | | States of the world | |
|---|---|---|---|
| | | Your party wins the election (p) | Your party does not win the election(1-p) |
| Available alternatives | To vote | | |
| | Not to vote | | |

Forcing you to make your assumption explicit is probably the **most important thing** that a formal model allows you to do: useful both for yourself and for the reader! It's a free psychoanalytic therapy session!

# The voter dilemma: to vote or not to vote?

| | | States of the world | |
|---|---|---|---|
| | | Your party wins the election (p) | Your party does not win the election(1-p) |
| Available alternatives | To vote | | |
| | Not to vote | | |

**Payoffs:**

Benefits: **Bi** (instrumental benefits if your party wins); **Be** (expressive benefits of voting)

Costs: **c** (the cost of voting)

# The voter dilemma: to vote or not to vote?

| | | States of the world | |
|---|---|---|---|
| | | Your party wins the election (p) | Your party does not win the election(1-p) |
| Available alternatives | To vote | Bi+Be-c | Be-c |
| | Not to vote | Bi | 0 |

**Payoffs:**

Benefits: **Bi** (instrumental benefits if your party wins); **Be** (expressive benefits of voting)

Costs: **c** (the cost of voting)

# The voter 2019 dilemma: to vote or not to vote?

| | | States of the world | |
|---|---|---|---|
| | | **Your party wins the election (p)** | **Your party does not win the election(1-p)** |
| **Available alternatives** | **To vote** | **Bi+Be-c** | **Be-c** |
| | **Not to vote** | **Bi** | **0** |

You will vote iff *Be>c (irrespective of p)*

# The voter dilemma: to vote or not to vote?

| | | States of the world | |
|---|---|---|---|
| | | Your party wins the election (p) | Your party does not win the election(1-p) |
| **Available alternatives** | **To vote** | **Bi+Be-c** | **Be-c** |
| | **Not to vote** | **Bi** | **0** |

How to interpret the results? **We do not know** what you are going to do (it depends on the values of the parameters!) but what **we do know** is that, according to our description of the situation, everything that increases *Be* (and/or decreases *c*) will also increase the probability that you will choose to vote. Moreover, this choice is *unaffected* by any policy promises made by your favorite party per-se (the impact of *Bi* via *p*)