Game Theory

2: Static games in normal form
and dominated strategies

Review of lecture one

- Main concepts of GT
- The theory of choice
  - Rational choosers (choice set, utility)
  - Utility function (maximization)
  - Choice under uncertainty (states of the world, lotteries, expected utility)
  - Risk preference

Interdependence

- In the theory of choice **uncertainty** in choice depends on **casual events** (the state of the world: natural and social)
- In GT uncertainty depends on **deliberate choices** of other individuals
- Then what an individual gets **depends** on everybody's intentional choice
- As a result, players interact and their results and payoffs are **interdependent**

Interdependence and probability

- As the choices of other people are **not casual but deliberate**, *probability* is not anymore the right concept to deal with when interactions are interdependent
- Expected utility calculus has to give place to some other conceptual device
- This device is **strategic reasoning** that game theory formalizes

Main concepts of strategic reasoning

- Players
- Rules of the game
- Strategy
- Payoffs

Players

- Players are assumed to be rational actors involved in strategic interaction
- **Remember**! They are not necessarily “enemies”, but simply interested to get their best personal utilitarian benefits
- In principle all motives (such as egoism, altruism, envy, love) can be included in their utility function

Rules of the game

- The rules say **what** each player is allowed to do, and **when** during the game
- They include
  - **Permitted moves**, i.e., the choice set given to every player whenever it is his turn to act
  - **Payoffs** that players receive at the end of the game
  - and…

Common Knowledge

- **Information** players have whenever they move
- In each game it must be clearly stated what is "**common knowledge**", i.e., which is the kind of information **available to all players**:
- ➢ the payoffs? the moves (and the sequence of moves)? the types of players?

Common Knowledge

- **Players' rationality** is (**generally**) always common knowledge, that is…
  - Each player believes that she is rational
  - Each player believes the other is rational
  - Each player believes the other believes that she is rational
  - Etc. up the nth-order

Strategy - I

- A strategy is a complete **plan of action** that a player has decided to play **before the game takes place**
- That is, a strategy is a complete **instruction guide** that takes account of every possible conditional development of the game

Strategy - I

- A strategy is not **therefore** the simple sequence of the moves played, but the **sequence of reasoning** considering any possible reaction of the player to any responses by other players
- When a player has defined a strategy it is **immaterial** whether she personally plays or assigns to an agent or a computer to play for herself

Strategy profile and payoffs

- A set of strategies, one for each player, is called a **strategy profile**
- In this sense, to a strategy profile corresponds a result (outcome), which assigns a **payoff** to each player, while each player's payoff depends on the **entire set** of strategies chosen by all players
- By definition of "strategy", when players have selected a strategy profile, the game is ended

Some notations

- Set of players**:** $I=\{1, 2, \ldots, i, \ldots, n\}$ (*i* is the generic player)
- Set of *i*'s strategies: S_i
- A single *i*'s strategy**:** $s_i \in S_i$
- A strategy profile: $S=\{s_1, s_2, \ldots, s_n\}$
- *i*'s payoff: $\pi_i(s_1, s_2, \ldots, s_n)$
  1. the payoff of a single player depends on the entire strategy profile
  2. payoff is another **word** for utility

Example

- Three players A, B, C
- Player A has two strategies available: a_1 and a_2
- Player B has three strategies: b_1, b_2, b_3
- Player C has two strategies available: c_1 and c_2
- Combining all available strategies leads to 12 possible strategy profiles: for instance (a_1,b_2,c_1)

Example

- All these profiles can be shortly written as: ***(111), (112), (121), (122), (131), (132), (211), (212), (221),(222), (231), (232)***
- To each profile a triple of payoffs correspond, one for each player:
  - for instance $\pi_A(132)$ correspond to what player A gets when he plays a_1, while B plays b_3 and C plays c_2

Strategies and moves

- The **solution** of a game is given ALWAYS in terms of strategies (more on this later), but the **rules** of the games are expressed through **moves** (i.e., who moves first? Second? Do they move together? Etc.)
- It is therefore important to know how to pass **from moves to strategies**

Example (first part)

- Two students, Ann and Bert, deciding to prepare math (M) or statistics (S)
- They may choose whether to **study together** (i.e., the same subject) or **not**
- Ann chooses first, then Bert chooses, observing what Ann did earlier
- **How many strategies** do they have?

Example (second part)

As Ann moves first she cannot make her decision dependent on Bert's; then **Ann has two strategies: (M) and (S) that coincide with her moves**

Bert can conceive more plans of action (strategies); they are:

1. play (M) whatever Ann's choice is: (M)vs.(M) and (M)vs.(S) ➔ **(MM)**
2. play (M) if Ann plays (M) and play (S) if Ann plays (S): (M)vs.(M) and (S)vs.(S) ➔ **(MS)**
3. play (S) if Ann plays (M) and play (M) if Ann plays (S): (S)vs.(M) and (M)vs.(S) ➔ **(SM)**
4. play always (S): (S)vs.(M) and (S)vs.(S) ➔ **(SS)**

The number of B's strategies is $2^2 = 4$ because there are 2^2 ways of responding to the 2 strategies of A with each of his 2 possible moves, while the moves available to B remains just two (M and S)!

Comment

- The preceding example is very simple but may help clarify:
  1. The very concept of strategy and its **difference** from the much simpler concept of move
  2. The importance of **information** while playing the game (if Bert does not know Ann’s choice, his strategies would be … how many?)
- Information is **so important** in GT that games are classified depending on the information available in them!

A taxonomy of games according to information

- **Perfect information games**: when players knows the moves played by the other players
- **Imperfect information games**: when players do not know the moves played by the other players
- **Complete information games**: when players know each others' payoffs (i.e. utility functions)
- **Incomplete information games**: when some players do not know some others' payoffs

Of course you can have any possible combination: games of perfect and complete information, games of imperfect and incomplete information, etc.

A taxonomy of games according to moves

- **Static games**: when players **move simultaneously**. Therefore, they are games of imperfect information (that can also be complete or incomplete) !!!
- **Dynamic games**: when players **move in sequence**. Therefore, they are games of perfect information (that can also be complete or incomplete) !!!

Of course you can have a game with both a static and a dynamic aspect!

Representation of games

In general:

- ✓ Static games require **normal or strategic form** representation (pointing out strategies)
- ✓ Dynamic games require **extensive form** representation (pointing out moves)

Sometimes the extensive form is convenient for static games, and vice versa normal form is convenient for dynamic games. We will discuss it later…

Static games in normal form

The normal (strategic) form of a game specifies:

- The **players** involved in the game
- The **strategies** (**here = moves. Why that???**) available to each player
- The **payoffs** to each player for every strategy profile

The normal form of a static games of **two players** is a **table** (**matrix**) where

- in the **rows** the strategies are numbered of one player
- in the **columns** the strategies of the other
- in the **cells** the payoffs are given

Example: Prisoners' dilemma

Two people are charged of a crime and interrogated separately by police

Each prisoner is told:

- If you confess and the other does not we promise you liberty (3) and a reward (1), and the other a fine (-2)
- If you both confess each of you will receive a lesser fine (-1)
- If neither confess, you both will be set free (no reward and no punishment)

Each player has therefore two moves that correspond to two strategies: "Confess" and "Stay silent"

Prisoners’ dilemma in normal form

- Liberty’s utility: 3
- Reward: 1
- Fine: -2
- Lesser fine: -1

		Second prisoner	
		Stay silent	Confess
First prisoner	Stay silent	**(3 , 3)**	**(−2 , 4)**
	Confess	**(4 , −2)**	**(−1 , −1)**

“Solving” the Prisoners’ Dilemma

- In the light of the game theoretical assumptions that we made, is there any way to **forecast** the two people choice of strategy?
- Equivalently: is there any reason to **anticipate** the cell of the matrix where the game will end, assigning to players the relative payoffs?
- For doing that, let’s introduce the concept of “**dominated strategy**”

Formal definition of a dominated strategy in a 2 players game

- In a two players game, *A* and *B*, let *x* and *y* be two strategies belonging to player *A*

➔ *x* is (**strictly**) dominated by *y* for player *A* if and only if

$$\forall s_B \in S_B \quad u_A(x, s_B) < u_A(y, s_B)$$

➔ Will you ever select a dominated strategy (irrespective if strictly or not) ?

Formal definition of a dominated strategy in a N players game

- Let $I = \{1, 2, \ldots, n\}$ be the set of players, and x and y be two strategies belonging to player i's set of strategy:

$$x, y \in S_i \ (i \in \{1, 2, \ldots, n\})$$

- Be s_{-i} a set of strategies, one for each player except i, so that $s = (s_i, s_{-i})$ is a strategy profile

➔ x is (strictly) dominated by y for player i if and only if

$$\forall s_{-i} \in S_{-i} \quad u_i(x, s_{-i}) < u_i(y, s_{-i})$$

Solution of the PD game

		Second prisoner	
		Stay silent	Confess
First prisoner	Stay silent	**(3 , 3)**	**(−2 , 4)**
	Confess	**(4 , −2)**	**(−1 , −1)**

Playing "Stay silent" gives to each player a worse payoff, **no matter how the other player behaves**

For this reason "Stay silent" is called **a dominated strategy**

Rationality assumption implies that **no player will ever play a dominated strategy**

Then **(Confess, Confess)** is the solution of the game!

Solution of the PD game

		Second prisoner	
		Stay silent	Confess
First prisoner	Stay silent	**(3 , 3)**	**(−2 , 4)**
	Confess	**(4 , −2)**	**(−1 , −1)**

What is strange about (Confess, Confess)?

As a solution it is **not socially efficient** (from the point of view of the two prisoners): both players **would be better off** if they kept quiet

Which are the implications of this result? More on this in the next lecture…

The general form of PD

		Player B	
		Cooperate (C)	Defect (D)
Player A	Cooperate (C)	(a_{CC}, b_{CC})	(a_{CD}, b_{DC})
	Defect (D)	(a_{DC}, b_{CD})	(a_{DD}, b_{DD})

In the previous example:
“stay silent”=“cooperate” **and** “confess=defect”
Must be (DC)>(CC)>(DD)>(CD) for both A and B

Exercise

Complete the following table with arbitrary numbers so that it represents a PD game

Must be (DC)>(CC)>(DD)>(CD) for both A and B

		Player B	
		(C)	(D)
Player A	(C)	(3 , 2)	(-2, …)
	(D)	(… , …)	(0, 1)