Game Theory

3: Nash equilibrium

Review of lecture two

- Interdependence in social action
- Main concepts in GT
  - players
  - strategies
  - payoffs
  - static games in normal (matrix) form
  - Prisoner dilemma
  - dominated strategy

“The battle of the sexes”

- A couple is choosing an evening entertainment
- Man prefers to go to a fight, and woman to a ballet
- Both prefer to go out together than to see alone the preferred entertainment
- Let’s represent the situation with a matrix

The BS game in normal form

		Woman	
		fight	ballet
Men	fight	**(2 , 1)**	**(0 , 0)**
	ballet	**(0 , 0)**	**(1 , 2)**

Solving BS

- In this case **no dominated strategy exists**
  - Man would choose “fight” if Woman chose “fight” but would choose “ballet” if Woman chose it
  - Woman would choose “ballet” if Man chose “ballet” but would choose “fight” if Man chose “fight”
- As a result, no solution is possible **by suppressing dominated strategies**
- **What to do then?**

Nash equilibrium

- An important solution concept for games is called a **Nash equilibrium**
- A Nash equilibrium is a **set of strategies** in a game (one for each player) such that no player has an incentive to unilaterally switch to another strategy, i.e., no player has an **incentive to change her mind** given what the other players are doing

Nash equilibrium (for two players)

- John Nash puts this idea in the following formal way
- A strategy profile $s^*=(s_A^*, s_B^*)$ of a two players game is a Nash equilibrium (NE) if and only if

$$\text{for every } s_A \neq s_A^* \text{ and for every } s_B \neq s_B^*$$

$$u_A(s_A^*, s_B^*) \geq u_A(s_A, s_B^*)$$

$$u_B(s_A^*, s_B^*) \geq u_B(s_A^*, s_B)$$

- When a NE is reached **no one** has incentive to change strategy

Nash equilibrium (for two players)

Saying differently: a strategy profile *(a*,b*)* is a NE if and only if:

$$u_A(a^*,b^*) \geq u_A(a, b^*) \ \underline{\text{and}}$$
$$u_B(a^*,b^*) \geq u_B(a^*,b)$$

where *a* and *b* may be either a_1 or a_2 and *b* may be either b_1 or b_2

Nash equilibrium (for two players)

- In a two player game a Nash equilibrium is a couple of strategies that are **mutual best response to each other's strategy**
- If one player plays a Nash strategy, the other player should play the corresponding strategy of the Nash profile (if she **does not want to decrease her payoff**), and vice versa

Nash equilibrium (for *n* players)

- A strategy profile $s^*=(s_1^*, s_2^*, \ldots, s_n^*)$ of a n players game is a NE if and only if

for every $i \in I$, and for every $s_i \in s$, $s_i \neq s^*$

$$\pi_i(s_{-i}^*, s_i^*) \geq \pi_i(s_{-i}^*, s_i)$$

where $-i$ is the set of all individuals except i

($-i=\{1, 2, \ldots, i-1, i+1, \ldots, n\}$)

Applying the formula to PD

		B	
		Stay silent (b_1)	*Confess* (b_2)
A	*Stay silent* (a_1)	(3,3)	(1,4)
	Confess (a_2)	(4,1)	(2,2)

The strategy profile (a_2,b_2) is a NE

One can verify that (a_2,b_2) is the only NE

$$u_A(a^*_2,b^*_2) = 2 > u_A(a_1,b^*_2) = 1\ ;$$

$$u_B(a^*_2,b^*_2) = 2 > u_B(a^*_2,b_1) = 1$$

Applying the formula to PD

		B	
		Stay silent (b_1)	*Confess* (b_2)
A	*Stay silent* (a_1)	(3,3) → ↓	(1,4) ↓
	Confess (a_2)	(4,1) →	(2,2)

The nice property of Nash equilibria is that they are **stable**! It **does not matter** from where you start, you always will converge to the NE of the game!

Dominated strategies and NE

- When a solution of the game can be find out through elimination of **dominated strategies** (as in PD) this solution is also a NE
- However NE may exist also when no dominated strategies exist (as in BS)
- Indeed most games have NE (often more than one)

Why looking for a NE?

- It represents a clear (i.e., unambiguous) **theoretical benchmark** against which it becomes possible to compare empirical data…
- …and from which you can derive hypotheses to be tested

The general meaning of PD game

- Let’s go back for example to the PD game
- Even thought the judge cannot convict on the robbery charge and it is in the suspects’ **collective interests** not to turn state’s evidence (i.e., to cooperate among themselves), it is likely that **both will confess**, claiming that the companion made him to do it. But **why**?
- It is only by exploring the **underlying** configuration of payments and its implication for identifying the anticipated strategy of the suspects that this becomes understandable

The general meaning of PD game

- This final outcome does not depend on the fact that the players are in some “metaphysical” sense bad or egoistic! It is also due to their **uncertainty**
- The criminals may well have promised each that if they were both caught they would keep quite. The problem is that these promises are **not credible**. Once they are caught, the criminals have a (dominant) strategy to talk

The general meaning of PD game

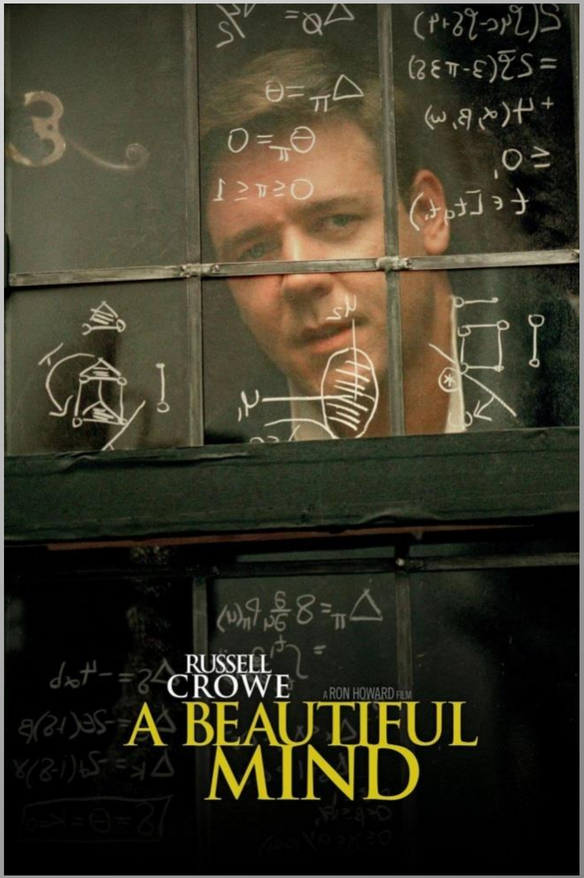
- As a result, **PD is a strategic scheme that points out that social cooperation is a difficult task**
- The absence of cooperation represents a **dilemma** – **individual rationality** leads players to an outcome that is an inferior one, i.e., both players agree on the fact that the same alternative outcome is a better one for them. Still they **cannot** reach it!
- Common interests are the mere **occasion** for cooperation. But the mere fact of common interest does not define nor determine behaviour

Applications of PD

- “*Politics is just a way to solve PD*”
- ✓ Nuclear arms race during the Cold War: strategies are “enhance” and “stop”
- ✓ Provision of public goods by citizens: strategies are “evade” and “pay taxes”
- ✓ International trade: strategies are: “rise trade barriers” and “do not rise”
- ✓ And…

Applications of PD

- http://www.youtube.com/watch?v=36M_wWShgMY

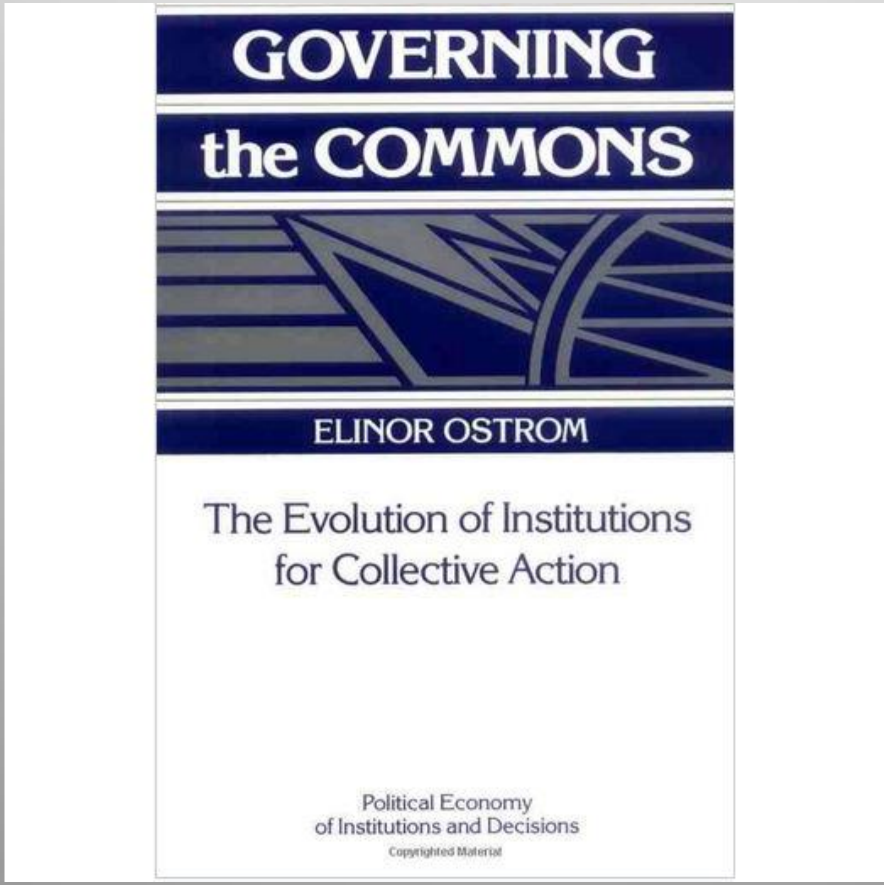


Applications of PD

- So cooperation is doomed to fail?
- Accepting the PD as an *analytical tool* does not **necessarily** implies that its predictions always turn into reality
- However, it forces us to understand when cooperation can become rational, that is, under which circumstances individuals find that the benefit of cooperation are enough to induce them to cooperate

Applications of PD

- More on this later (if we have time…)



Applying the NE formula to BS

		B	
		b_1	b_2
A	a_1	(2,1)	(0,0)
	a_2	(0,0)	(1,2)

The strategy profile (a_1,b_1) is a NE

$$u_A(a^*_1,b^*_1)=2 > u_A(a_2,b^*_1)=0 \ ;$$

$$u_B(a^*_1,b^*_1)=1 > u_B(a^*_1,b_2)=0$$

One can verify that also (a_2,b_2) is a NE while the other two profiles are not

Example

		B	
	b_1	b_2	b_3
a_1	3,8	9,6	1,9
A a_2	5,5	7,2	6,3
a_3	4,7	2,8	8,3

Example

			B	
		b_1	b_2	b_3
	a_1	3,8	9,6	1,9
A	a_2	5,5	7,2	6,3
	a_3	4,7	2,8	8,3

Both conditions are satisfied only for the result (5,5), so that (a_2,b_1) is the only NE of the game

$u_A(a^*_2,b^*_1)=5 > u_A(a_1,b^*_1)=3; u_A(a_2,b^*_1)=5 > u_A(a_3,b^*_1)=4$

Exercise

		B		
		b_1	b_2	b_3
A	a_1	3 , 4	2 , 5	0 , 1
	a_2	8 , 2	7 , 1	2 , 3
	a_3	4 , 5	6 , 6	1 , 7

Find all NE of the game (there are **two ways** for doing it…)

Exercise

		B		
		b_1	b_2	b_3
A	a_1	3 , 4	2 , 5	0 , 1
	a_2	8 , 2	7 , 1	2 , 3
	a_3	4 , 5	6 , 6	1 , 7

Find all NE of the game

Types of games and their solutions

	Complete information	**Uncomplete information**
Static games (imperfect information)	Nash equilibrium	Bayesian Nash equilibrium
Dynamic games (perfect information)	Subgame perfect equilibrium	Perfect Bayesian equilibrium

Refinements of Nash equilibrium