Game Theory

4: Nash equilibrium in different games and mixed strategies

Review of lecture three

- A game with no dominated strategy: “The battle of the sexes”
- The concept of Nash equilibrium
- The formal definition of NE
- How to find NE in matrix games

A variety of games

- Let’s explore some typical game-structure we can have out there (in normal form)

Which side of the road?

		Mr. Red	
		LEFT	RIGHT
Mr. Green	LEFT	(1 , 1)	(-1 , -1)
	RIGHT	(-1, -1)	(1 , 1)

- You have two NE: (LEFT, LEFT) and (RIGHT, RIGHT)
- This game is a **pure coordination** one
- In a coordination game the problem is to **find out the way to get the mutual benefit by coordinating their actions** (i.e., choosing the same strategy)
- How to do that?

Focal points

- The identification of the NE in such instance asks for a **richer knowledge the external environment**
- For games of coordination this can be done by finding some elements of interaction that **appears in some way as prominent to players** (due to culture, past actions, etc.)
- These aspects are called **focal points**

Examples (T. Schelling 1960)

- “*People can often concert their intentions or expectations with others if each knows that the other is trying to do the same*”
- Focal points **impose themselves** on the players’ attention for reasons that the formal theory overlooks
- To be a true focal point, not only should it be **obvious** to each, but everyone should know that it is obvious to each, and everyone should know that…; in other words, its **obviousness should be common knowledge**

Examples (T. Schelling 1960)

- The existence of focal point is often a matter of coincidence, and creating them where none exist is basically an art that requires a lot of attention to the historical and cultural context of a game, and not merely its mathematical description

Examples (T. Schelling 1960)

- Name a city of UK. If you **all** name the same, you win a prize
- You are to meet somebody in Milan for a reason that you both consider very important. You were not been told where to meet and you cannot communicate. Guess where to go…
- But the environment in which taking decisions of course matters a lot…

Examples (T. Schelling 1960)

- Two players are asked to write down, simultaneously and independently, the share that each wants for a total prize of $100
- If the amounts that they write down add up to $100 or less, each player is given what she wrote. If the two add up to more than $100, neither gets anything
- For any *x*, one player writing x and the other writing (100-*x*) is a NE
- Thus the game has an almost infinite range of NE…still what would you expect?
- 50:50 emerges as a focal point (a social norm of equality or fairness)

Examples (T. Schelling 1960)

- But once again, this is **contextually driven**
- Consider a situation in which one player is a woman from an **egalitarian society** who belivies that 50:50 is obvious and the other is a man from a p**atriarchal society** who believes it is obvious that a man should get three times as much as a woman
- Then each will do what is obvious to her/him, and they will end up with nothing, because neither's obvious solution is obvious as common knowledge to both

Other possible games out there!
How to depict the nuclear arms race between US and USSR? (first case)

- Both US and USSR have two strategies: “ARM” and “REFRAIN”
- Suppose both US and USSR care only for military supremacy

➔(A vs. R)>(R vs. R)>(A vs. A)>(R vs. A)

Tough superpowers arms race

		USSR	
		REFRAIN	ARM
US	REFRAIN	(3 , 3)	(1 , 4)
	ARM	(4 , 1)	(2 , 2)

- By choosing ARM (leading to (2,2) both players are worse off than if they could reach an arm control agreement, leading to (3,3)
- However this outcome is unstable (it is not a Nash Equilibrium)
- The only NE is (ARM,ARM) leading to (2,2) ➔ the game is a PD

How to depict the nuclear arms race between US and USSR?(second case)

- US and USSR have the same two strategies: “ARM” and “REFRAIN”
- But their leadership care also for military expenditures that reduce people’s standard of living
- However security is **now** worth more than expenditures

➔(R vs. R)>(A vs.R)>(A vs. A)>(R vs. A)

➔ this situations creates **another particular type of game**

Mild superpowers arms race

		USSR	
		REFRAIN	ARM
US	REFRAIN	(4 , 4)	(1 , 3)
	ARM	(3 , 1)	(2 , 2)

- Two NE (ARM,ARM) and (REFRAIN,REFRAIN)

Assurance game

		USSR	
		REFRAIN	ARM
US	REFRAIN	(4 , 4)	(1 , 3)
	ARM	(3 , 1)	(2 , 2)

- (REFRAIN,REFRAIN) is **better for both** but difficult to reach
- If one player has reason to think that the other chooses ARM, it too will choose ARM
- To choose REFRAIN a player (*more than a focal point…*) needs the assurance that the other will do the same ➔ **assurance game**
- In the case of superpowers this assurance would have been a mutual control…
- …however they never accepted it: the role of (mis)perceptions! Sometimes «**pessimism breads pessimism**»

Back to Cold War…

- Imagine that you **empirically observe** (ARM, ARM) (the actual NE since 60s till half of 80s…)
- Observing something **can be** however **quite misleading**…

What you see, it's not what you think you have seen…

Back to Cold War…

- A NE of (ARM, ARM) could be due either to a PD or to an Assurance Game given the presence of misperceptions, i.e., **2 completely different games**!
- This matters a lot! If the underlying game is (was?) a PD there is not way to change the NE! Optimism (for example about the incentive of the other player to play REFRAIN) would never change the (ARM, ARM) situation
- However, if the underlying game is (was?) an Assurance game…any effort to change the perception of each other player would have matter a lot!!! **Which was the real Cold War strategic interaction?**

How to depict the nuclear arms race between US and USSR? (third case)

- Superpowers acknowledge the situation has become dramatic (the Cuban crisis?)
- Both assume having two strategies: send an ULTIMATUM or RETRAIT
- Double U brings about a nuclear conflict (the worst case for both)
- The best result is to send U when the other plays R
- The second result is the double R
- R against U is the third result

➔(U vs. R)>(R vs. R)>(R vs. U)>(U vs. U)

Ultimatum game

		USSR	
		R	U
US	R	(2 , 2)	(1 , 3)
	U	(3 , 1)	(0 , 0)

- Two NE: (U,R)➔ (3,1) and (R,U) ➔ (1,3)
- This game is also called a **chicken game**: people do not coordinate on the **same** strategy!
- Which one of the two NE will be chosen depends on the availability of possible **strategic moves** (i.e., **credible pre-commitments**)

Strategic moves

- A player may take an **initiative** that influences the other player's choice in a way favorable to the first one
- One can **constrain the opponent's choice** by constraining one's own behavior in a CREDIBLE way: less "freedom" can in fact gives you a better payoff!!!
  - Bert can arrive home with the tickets for "fight" so that the choice "ballet" is implicitly cancelled
  - A military commander can order his guard to burn the bridge of the river just passed so that his army knows that can never retreat and must fight fiercely (*Sun Tsu, The art of war*)

Strategic moves

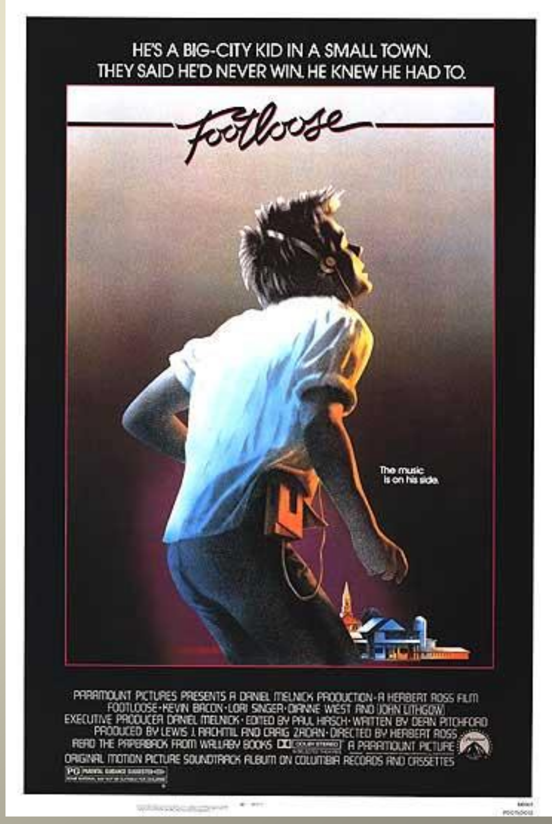
- Of course, if a strategic move is employed, this removes by definition the “imperfection” in the information setting of the game. Why?
- Cause all players will know now for sure what the player who has access to the strategic move will do
- The identification of the NE in such instance asks however once again for a **richer knowledge the external environment**

Back to cinema!

A Chicago teenager called Ren moves to a small city in Iowa. Ren's love of dancing and partying causes friction with the community. Much of the movie centers on the competition between Ren and the local tough guy named Chuck

At one stage Chuck challanges Ren to a "**tractor face-off**"

In this face-off Ren and Chuck have to drive tractors directly at each other. Whoever swerves out of the way first is considered a "chicken"



Let's represent the game

		Chuck	
		Straight	Swerve
Ren	Straight	(,)	(,)
	Swerve	(,)	(,)

Let’s represent the game

		Chuck	
		Straight	Swerve
Ren	Straight	(1, 1)	(4, 2)
	Swerve	(2, 4)	(3, 3)

Cinema time!

Strategic moves

- Finding a strategic move is one of the most difficult thing to achieve
- Sometimes this quest is made more easy by the existence of a “reputation” that is considered valuable to preserve

Strategic moves

- Fonzie does not need to fight to prove he is tough, cause everyone knows that he is tough, so he does not need to fight at all!

Strategic moves

- Until he finds someone that does not know Fonzie's reputation…someone from another planet, of course!

"Matching pennies"

– Two players: A and B own a coin each, turned secretly on head or tail

– Confronting coins, if both show the same face A takes both; otherwise B takes both

		B	
		head	tail
A	head	(1 , −1)	(−1 , 1)
	tail	(−1 , 1)	(1 , −1)

A *zero-sum game* (i.e., a game in which one player prefers a coincidence of actions and the other prefers the opposite) with no NE. What to do?

"The marital infidelity game"

- Two players: Husband and Wife
- Two strategies available to each of them: Husband (Faithful or Cheat) Wife (Monitor or Do not monitor) [or viceversa…]. What about the payoffs?

		Wife	
		Monitor (M)	Do not monitor (D)
Husband	Faithful (F)	(1 , 1)	(1 , 2)
	Cheat (C)	(0 , 2)	(2 , 1)

No NE!!! What to do?

Mixed strategy

- Every finite game (having a finite number of players and a finite strategy space) has at least one NE (in pure OR in mixed strategies)
- **A mixed strategy for a player is a probability distribution over her (pure) strategies**

Mixed strategy

- In the previous example: A (1/2, 1/2) is a possible mixed strategy in which head is played with probability=1/2 by player A and the same tail. Other possible mixed strategies can be: (2/3, 1/3), (1/4, 3/4) and so forth
- Note that a mixed strategy includes also **all pure strategies** (when the probability of a strategy is = 1 and the probability for the other strategy is = 0, i.e., A (1, 0))

Mixed strategy

- What is a **mixed-strategy Nash Equilibrium (MSNE)**?
- A MSNE is a profile of MS $M^* \epsilon$ **M** such that $u_i(M_i^*, M_{\_i}^*) \geq u_i(M_i, M_{\_i}^*)$ $\forall$ *i* and $M_i \epsilon$ **M**
- How to estimate a MSNE?
- Let's guess that A mixes between H and T. If this strategy is optimal for A (in response to the other player's strategy), then it must be that the **expected** payoff from playing H equals the **expected** playoff from playing T. Why that?
- Otherwise, player A **would strictly prefer** to pick either H or T (i.e., playing a pure strategy)

Mixed strategy

- But how can player A’s strategies H and T yield the **same expected payoff**?
- It must be that **player B’s behavior generates this expectation** (because if B plays a pure strategy, then A would strictly prefer one of its strategies over the other…but then also B would prefer to change her pure strategy and so on…)
- Let’s see how…

Mixed strategy

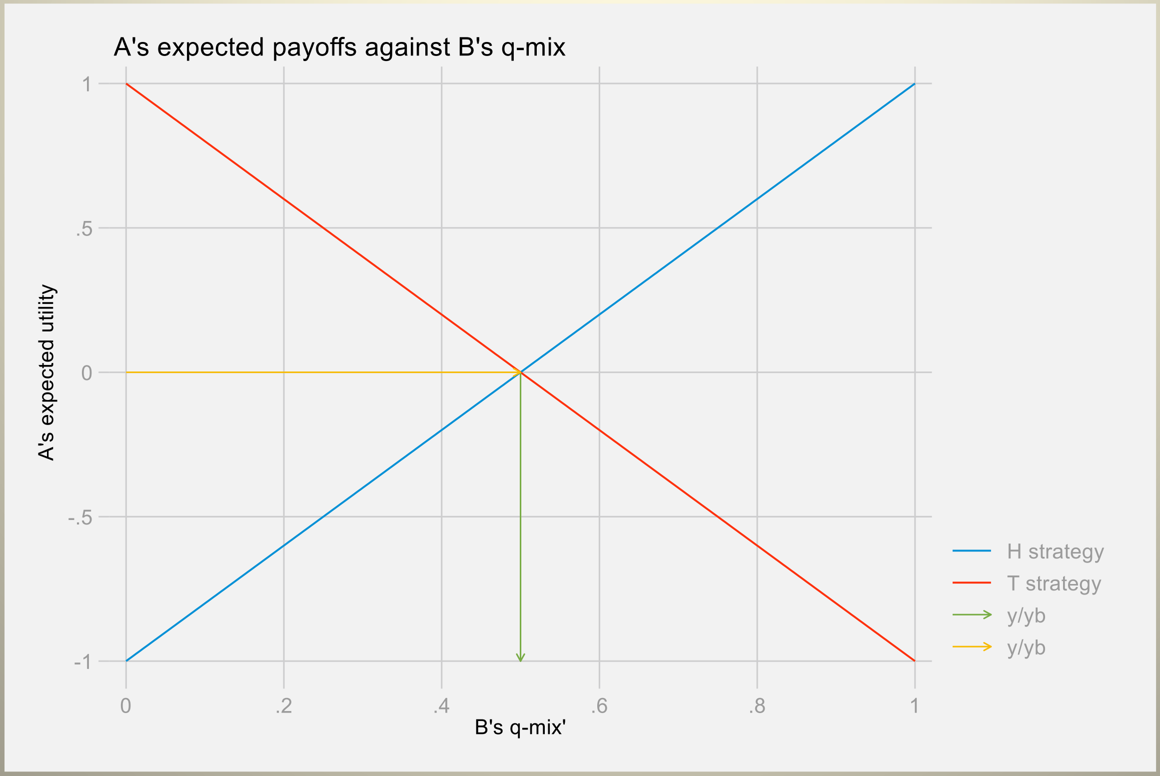
- Let’s call p the probability for A to play “head” and 1−p her probability to play “tail”
- Let’s call q the probability for B to play “head” and 1−q his probability to play “tail”
- So how to proceed?

"Matching pennies"

			q	1-q
			B	
			head	tail
p	**A**	head	(1 , −1)	(−1 , 1)
1-p		tail	(−1 , 1)	(1 , −1)

The "calculus" way

- $EU_A(H|q) = q-1+q = 2q-1$
- $EU_A(T|q) = -q+1-q=1-2q$
- $EU_A(H|q) = EU_A(T|q)$ implies $q=1/2$



The “calculus” way

- $EU_A(H|q) = q-1+q = 2q-1$
- $EU_A(T|q) = -q+1-q=1-2q$
- $EU_A(H|q) = EU_A(T|q)$ implies $q=1/2$
- Similarly:
- $EU_B(H|p) = -p+1-p = 1-2p$
- $EU_B(T|p) = p-1+p=2p-1$
- $EU_B(H|p) = EU_B(T|p)$ implies $p=1/2$
- The mixed strategy profile ((1/2, 1/2), (1/2, 1/2)) or (p,q)=(1/2, 1/2) is a MSNE

The “calculus” way

- Given player B’s mixed strategy (1/2, 1/2), player A’s mixed strategy (1/2, 1/2) is a best response, and vice-versa: i.e., **you have an equilibrium**!!!

The tricky aspect

- However note an interesting fact…
- **Every strategy** is a best response for player A, **given player B's mixed strategy in equilibrium**: i.e., (3/4, 1/4), (0, 1), (1, 0), etc.
- How is that? Check yourself!

The tricky aspect

- In this sense, if player A changes his strategy, given player B's mixed strategy, it **does not worse** his situation, that is, each player is indifferent between her pure strategies, or indeed between any mixture of them, so long as the other player is playing her correct (equilibrium) mix
- In a **pure NE**, on the contrary, if you deviate from your equilibrium strategy, you **always worse** your situation

The tricky aspect

- As a result: a MSNE is a NE only in a **weak sense** compared to a pure NE…
- **…still it is an equilibrium**, i.e., the solution to a strategic interdependent situation (and in some cases, the **only** solution available…) [i.e., if A plays something else than its mixed strategy in equilibrium, then B will have an incentive to change its strategy as well, and so on…**no equilibrium is reached**!]

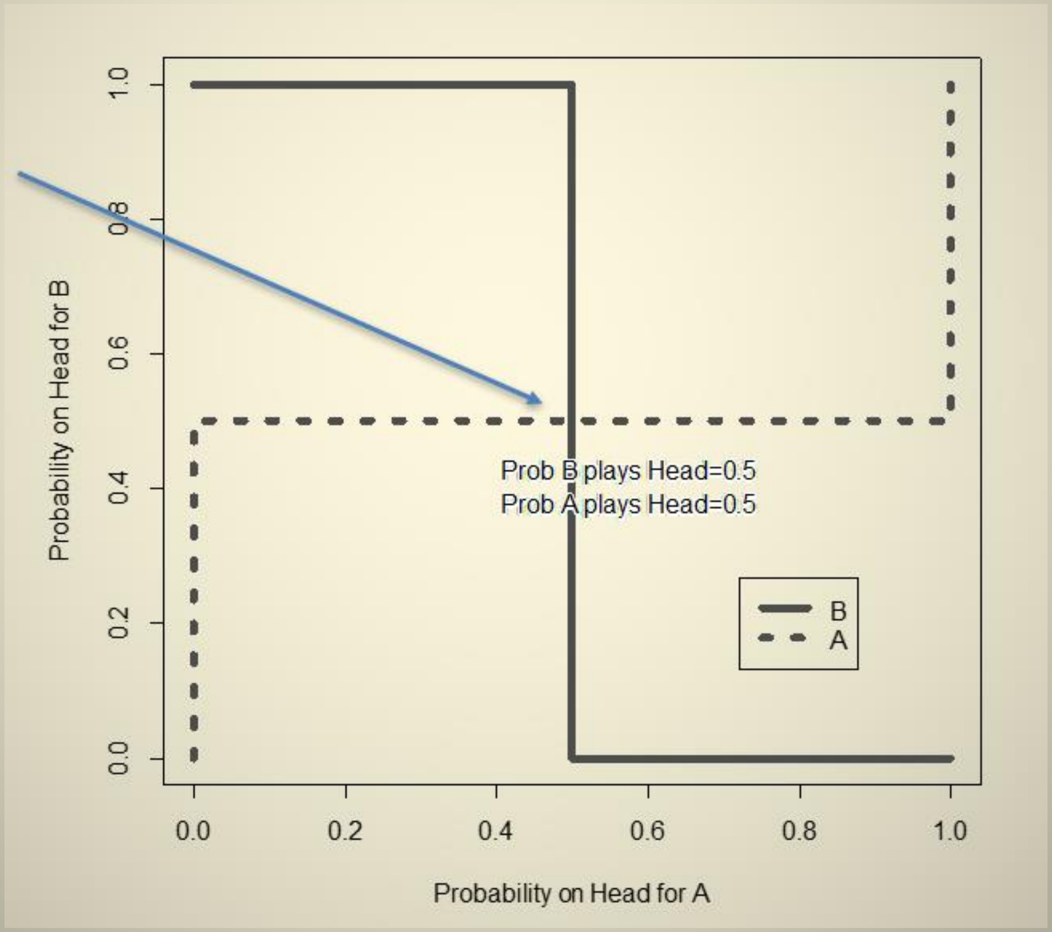
A "graphical" way

- Note that looking for a MSNE entails an interesting new twist: we look for a mixed strategy for one player that makes the **other** player indifferent between her pure strategies. This is the **best** method of calculating MSNE
- A **graphical way** to look at a MSNE…

Let's plot the Best Response curves!

A “graphical” way for the matching pennies game

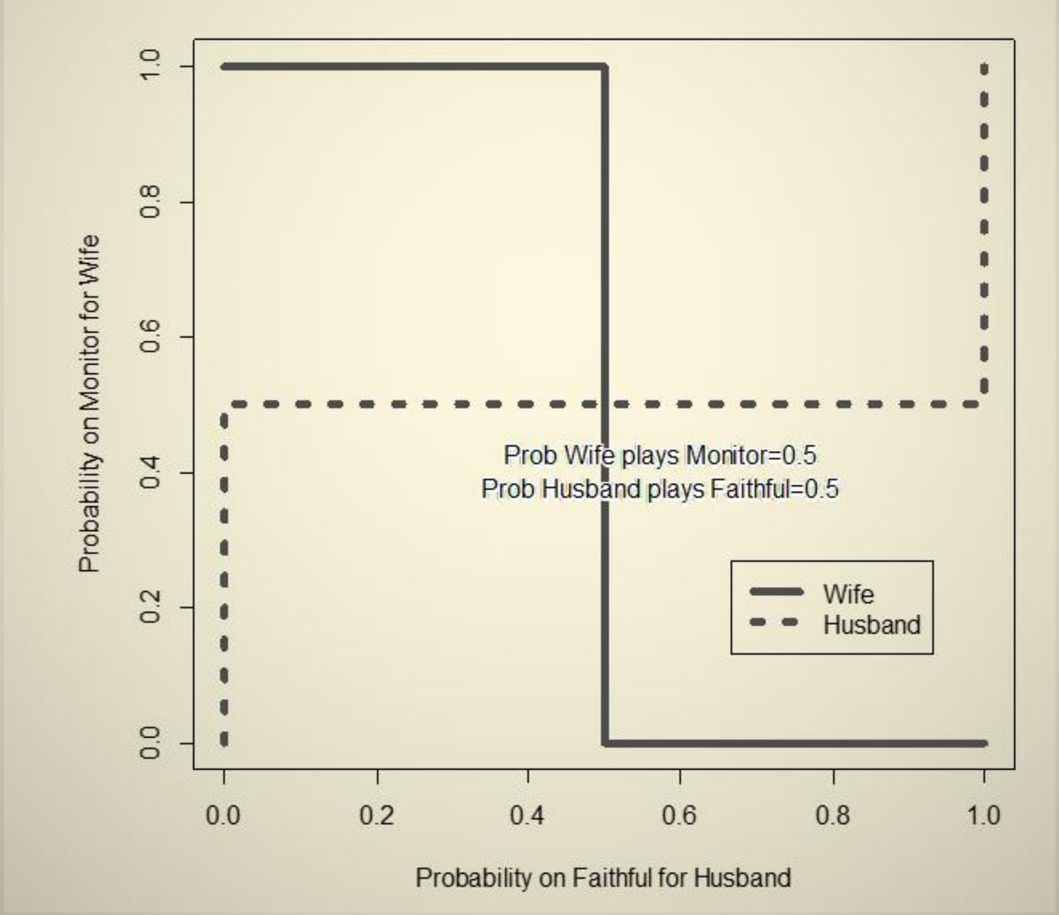
There is only one point (i.e., the MNE!) in which the two curves intersect



"The marital infidelity game"

		q →	1-q →
		Wife	
		Monitor (M)	Do not monitor (D)
p → **Husband**	Faithful (F)	(1 , 1)	(1 , 2)
1-p →	Cheat (C)	(0 , 2)	(2 , 1)

A "graphical" way for the marital infidelity game



An interpretation of MSNE

- **Repeated** game interpretation: the probabilities identified by a MSNE correspond to the **frequencies** of times that each strategy is played by each player **over time** in equilibrium
- **Evolutionary** game interpretation: the probabilities identified by a MSNE correspond to the **percentage** of players playing each pure strategy in a given population in equilibrium

An interpretation of MSNE

- Remember further the alternative definition we gave to NE in terms of **beliefs** - a set of strategies, one for each player, such that (1) each player has ***correct beliefs*** about the strategies of the others and (2) the strategy of each is the best for herself, ***given her beliefs*** about the strategies of the others
- With a mixed NE, we need to give a corresponding reinterpretation: perhaps players can be **unsure** about what others might be doing, and therefore may have **uncertain** beliefs that in turn lead her to be **unsure** about how she should act. This can lead to *mixed-strategy equilibria*

An interpretation of MSNE

- However, in such instance players will have **uncertain but still correct beliefs** (at least in equilibrium!)
- There is in fact a difference ***between being unsure*** and ***having incorrect beliefs***
- For example, the Husband cannot be sure of what the Wife is choosing on any one occasion. But he can still have correct beliefs about Wife's mixture (i.e., the probabilities with which she will choose between her two pure strategies)

An interpretation of MSNE

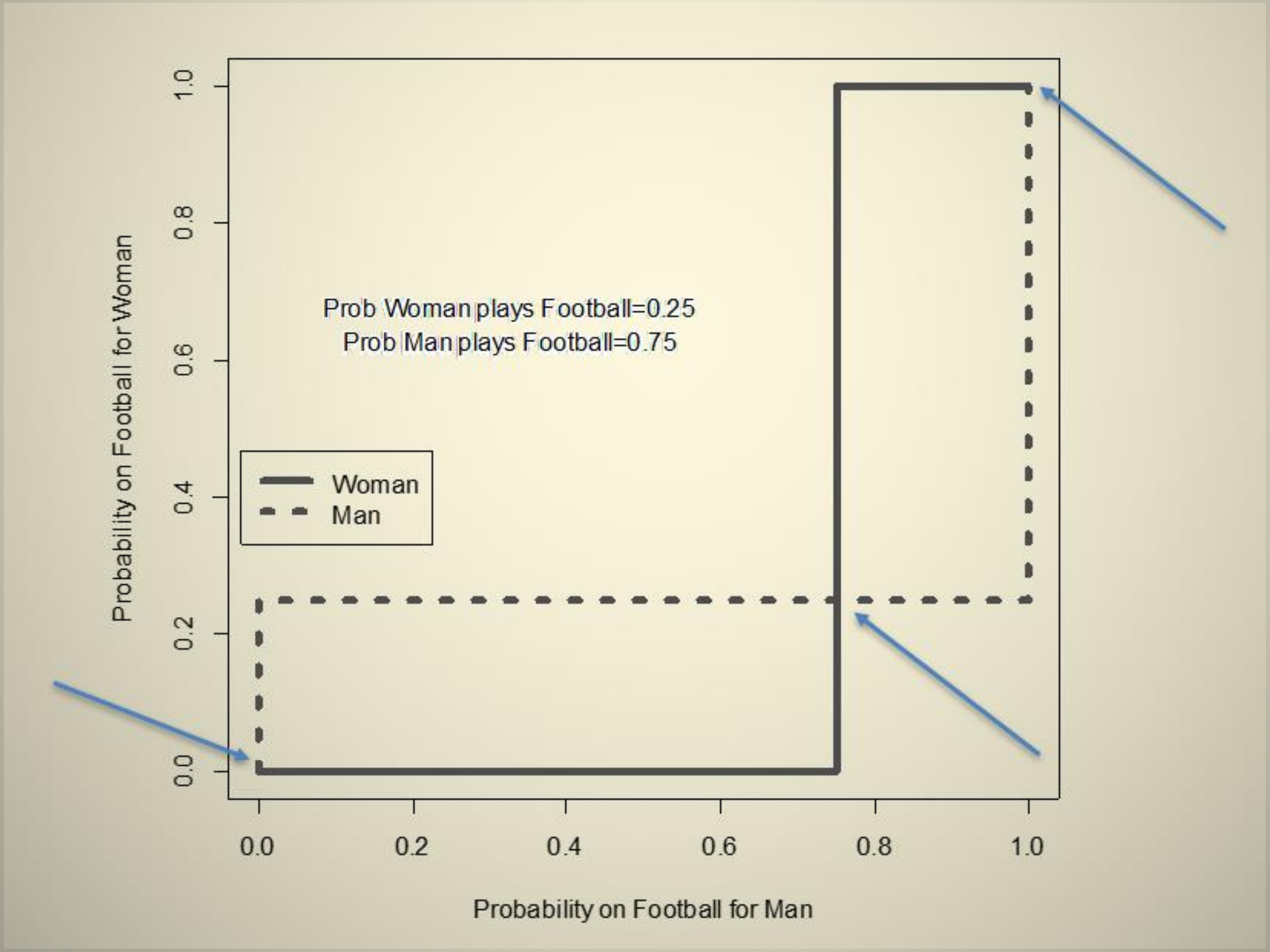
- Having **correct belies about mixed actions** means knowing or calculating or guessing the correct probabilities with which the other player chooses from among her underlying pure actions
- That is, in a Mixed NE, players' beliefs, *although uncertain*, will **be correct in equilibrium**
- From here an alternative and equivalent way to define a NE in terms of beliefs: each player forms beliefs about the probabilities of the mixture that the other is choosing and chooses her own best response to this. A mixed NE occurs **when the beliefs are correct**

The Battle of the Sexes reprise

		q	1-q
		Woman	
		Football	Opera
p	**Man** Football	(3 , 2)	(1 , 1)
1-p	Opera	(0 , 0)	(2 , 3)

- Man and Woman like each other, but Man of course likes football more than Opera…
- They have too coordinate their behavior…
- There are two pure NE and one MSNE
- Find them!
- Compared to a pure NE, a MSNE is less stable…

A “graphical” way for the Battle of the Sexes



Addendum

- Note that up to know, in equilibrium, the mixed strategies played by the two players are a complement to each other (i.e., they sum to 1)
- This should not to be the case! Solve the game below!

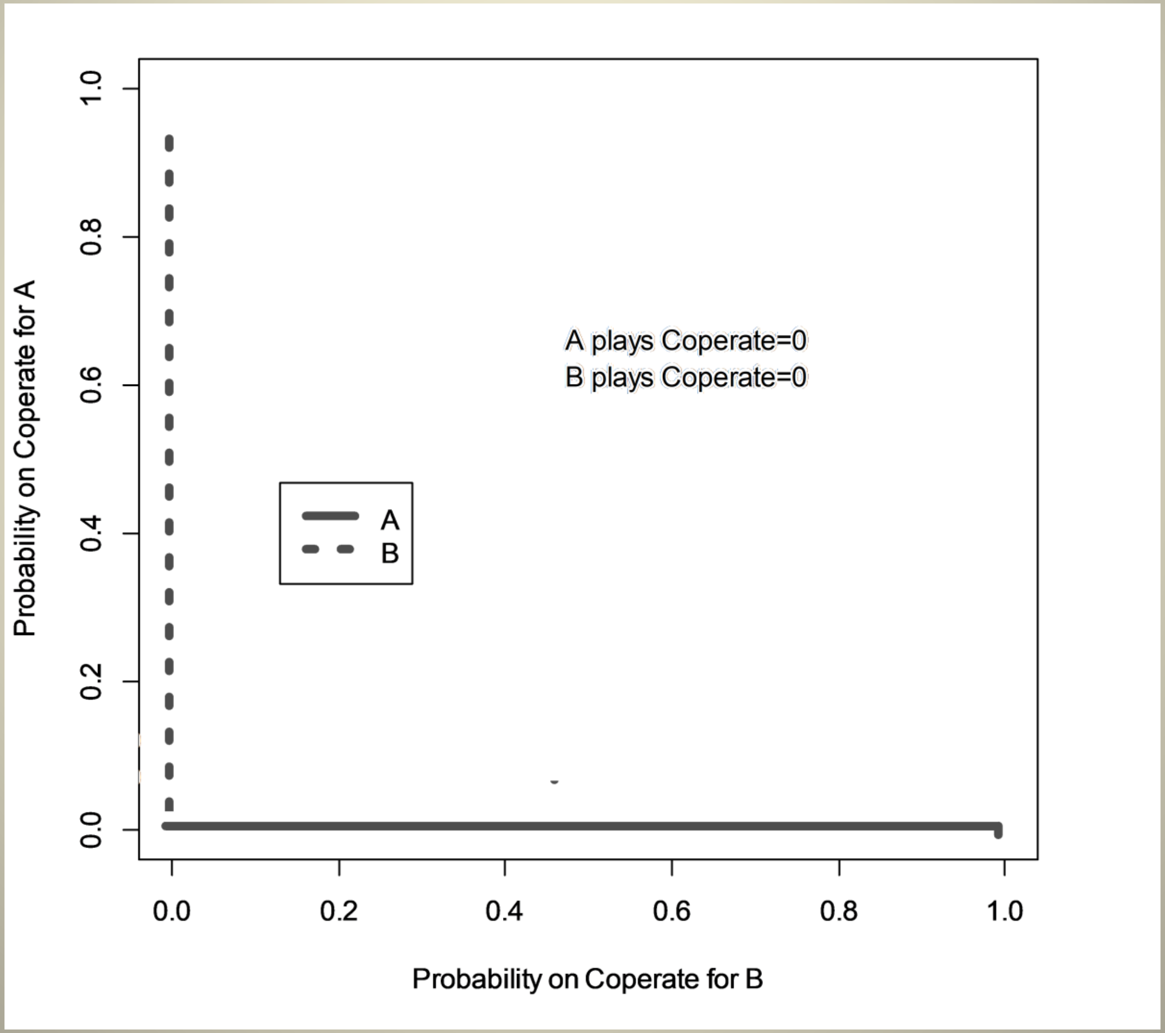
		Woman: Football (q)	Woman: Opera ($1-q$)
Man	Football (p)	(30 , 2)	(1 , 1)
	Opera ($1-p$)	(0 , 0)	(2 , 3)

A MSNE in a PD?

			q	1-q
			Player A	
			Cooperate	Defect
p	**Player B**	Cooperate	(3 , 3)	(1 , 4)
1-p		Defect	(4, 1)	(2 , 2)

- Can we have a MSNE in a PD? Yes or No? And why?

A "graphical" way for the PD



And with 3 strategies?

Goalkeeper column probabilities: ql (Left), 1-ql-qr (Center), qr (Right)

Kicker row probabilities: pl (Left), 1-pl-pr (Center), pr (Right)

		Goalkeeper		
		Left	Center	Right
Kicker	Left	45, 55	90, 10	90, 10
	Center	85, 15	0, 100	85, 15
	Right	95, 5	95, 5	60, 40

- Goalkeeper vs. Kicker – a zero sum game!
- In the cells probability of scoring (for the kicker) and of avoiding the goal (for the goalkeeper)
- How to solve the game? With our usual approach!

And with 3 strategies?

- EU_G(Left|pl, pr) = 55pl+15(1-pl-pr)+5pr
- EU_G(Center|pl, pr) =10pl+100(1-pl-pr)+5pr
- EU_G(Right|pl, pr) = 10pl+15(1-pl-pr)+40pr

For having a MSNE, EU_G(Left|pl, pr) = EU_G(Right|pl, pr), that is:

45pl=35pr, that is: pr=9/7pl

EU_G(Center|pl, pr) = EU_G(Right|pl, pr), that is:

-85pl-120pr+85=0, that is -595pl-1080pl+595=0,

That is: pl=0.355

As a result pr=3.195/7=0.456

As a result pc=(1-0.355-0.456)=0.189

And with 3 strategies?

- EU_K(Left|ql, qr) = 45ql+90(1-ql-qr)+90qr
- EU_K(Center|ql, qr) = 85ql+85ql
- EU_K(Right|ql, qr) = 95ql+95(1-ql-qr)+60qr

For having a MSNE, EU_K(Left|ql, qr) = EU_K(Right|ql, qr), that is:

-45ql-5+35qr=0, that is: ql=7/9qr-1/9

EU_K(Center|ql, qr) = EU_K(Right|ql, qr), that is:

17ql+24qr=19, that is 119qr-17+216qr=171,

That is: qr=0.561

As a result ql=2.927/9=0.325

As a result qc=(1-0.561-0.325)=0.114

And with 3 strategies?

MSNE=(pl, pc, pr) (ql, qc, qr)=(0.355, 0.189, 0.456) (0.325, 0.114, 0.561)

- ✓ The kicker's payoff from any of his pure strategies when played against the goalkeeper's equilibrium mixtures is 75.4 (the same one of the goalkeeper)
- ✓ The Kicker does better with his pure Right than his over Left (a left-footed?). Therefore the kicker chooses his Right with greater probability and, to counter that, the goalkeeper chooses Right with the highest probability, too. However, the kicker does not choose his pure-strategy right; if he did so, the goalkeeper would then choose Right too, but then, etc., and no equilibrium!

The World War I game (Homework)

- Consider the following scenario
- The **British** are deciding whether to attack **Germany** at the Somme river in France or to attack Germany's ally Turkey at Constantinople. The Somme is closer to German territory so a big victory there will end the war sooner that a breakthrough against Turkey
- The Germans must decide whether to concentrate their defensive forces at the Somme or bolster Turkey
- If the attacks comes where the defense is strong, the attack will fail. If the attack happens where the defense is weak, the attackers win

The World War I game (Homework)

- Assume that British preferences are given by u_B(victory at the Somme) = 2 > u_B (Victory in Turkey)=1 > u_B (losing either place) = 0 and that the preferences of the Germans are given by u_G (successful defense) = 2 > u_G (lose in Turkey) = 1 > u_G (lose at the Somme) = 0
- Further assume that the British strategy space is (attack the Somme, attack Turkey) and that the Germany strategy space is (defend the Somme, defend Turkey)
- So: a) represent this game in Matrix form; b) find all the pure strategy and mixed strategy NE of this game, using both methods discussed (i.e., including also drawing best reply correspondences)

Playing games with R

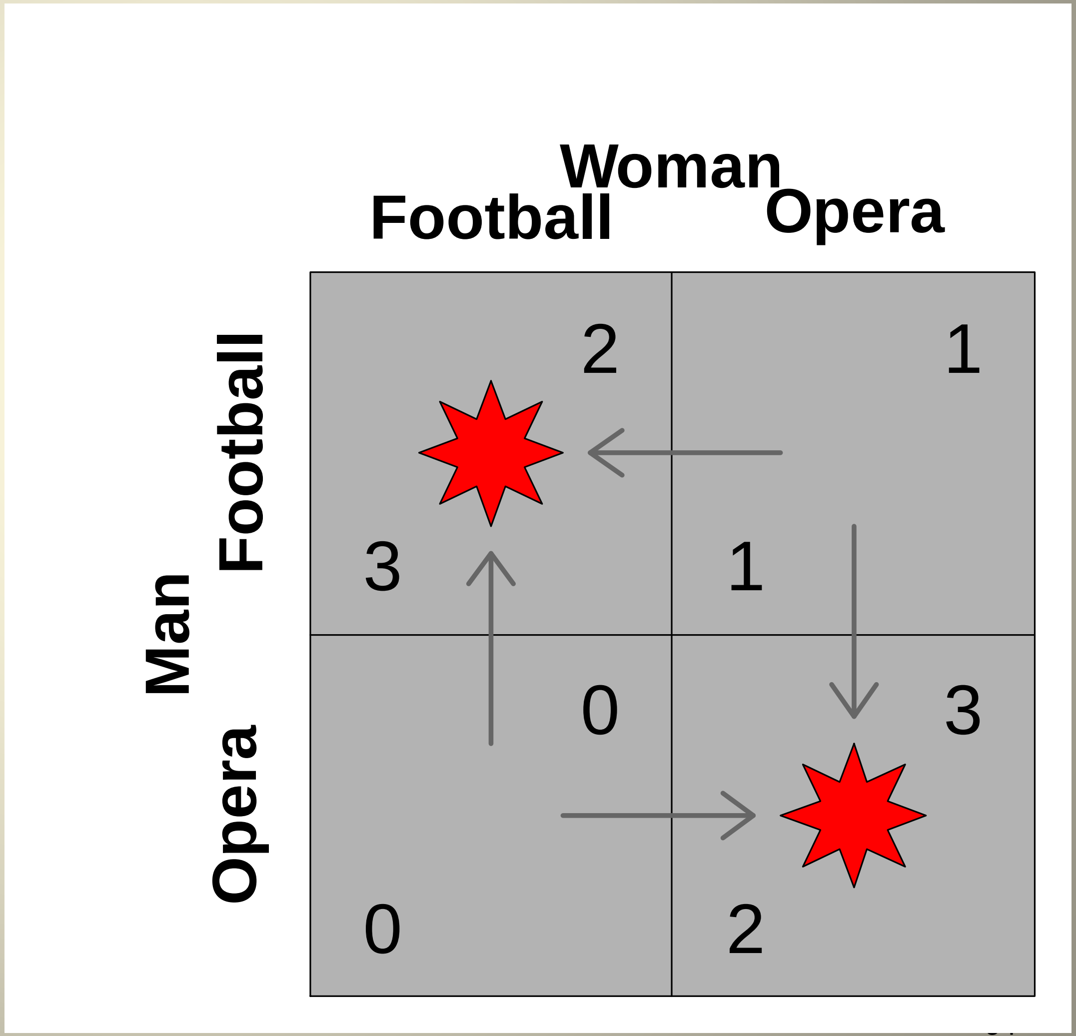
- A great package to run (and solve) static (and dynamic) games of complete information: **`hop`**
- http://www.macartan.nyc/games/normal-form/
- It also runs MSNE with graphs!
- How to install **`hop`** in R?

```
devtools::install_github("macartan/hop")
```

Playing games with R

- Some examples:

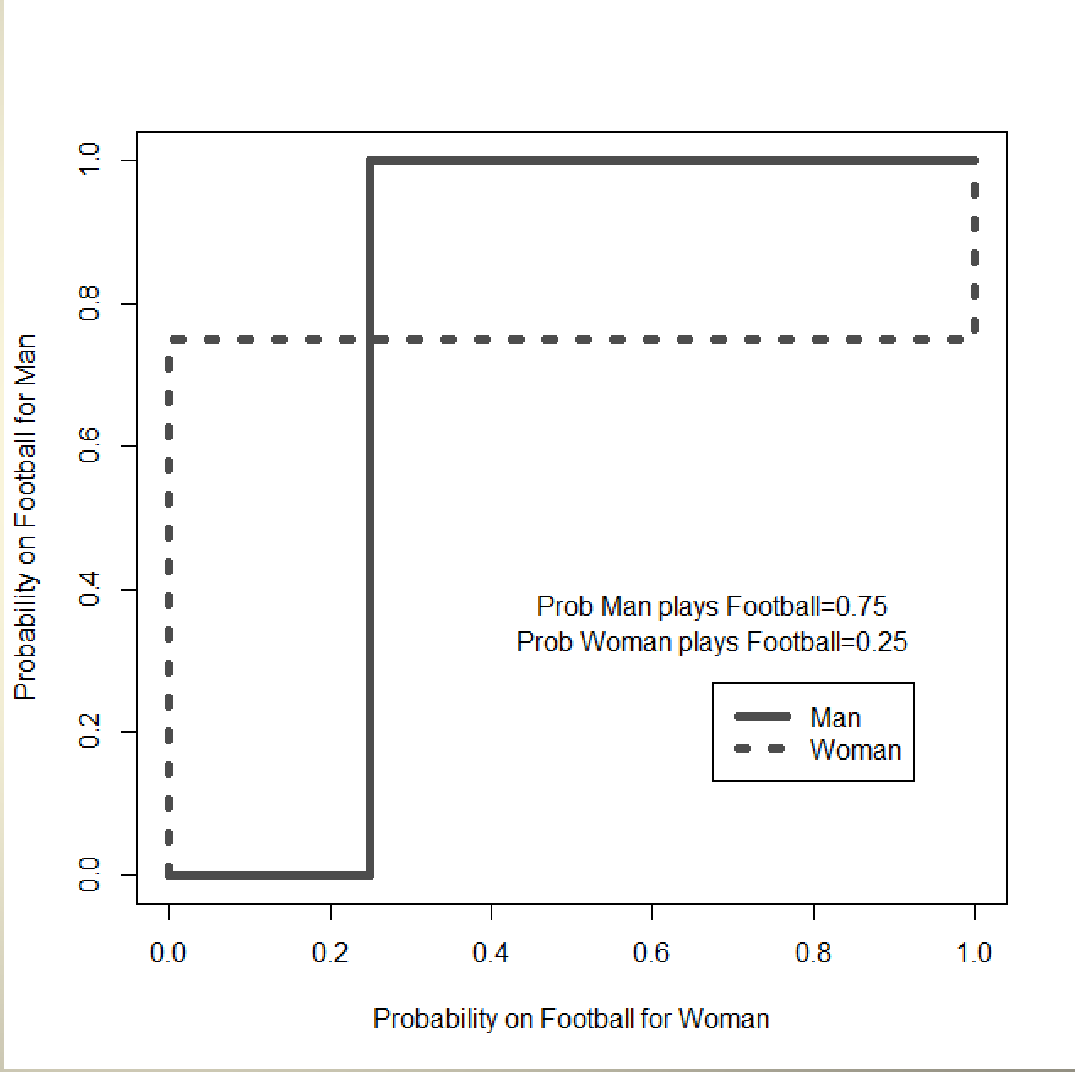
```
gt_bimatrix(X = matrix(c(3, 0, 1, 2), 2),
Y = matrix(c(2, 0, 1, 3), 2),
P1 = "Man",  P2 = "Woman",
labels1 = c("Football", "Opera"))
```



Playing games with R

- Some examples:

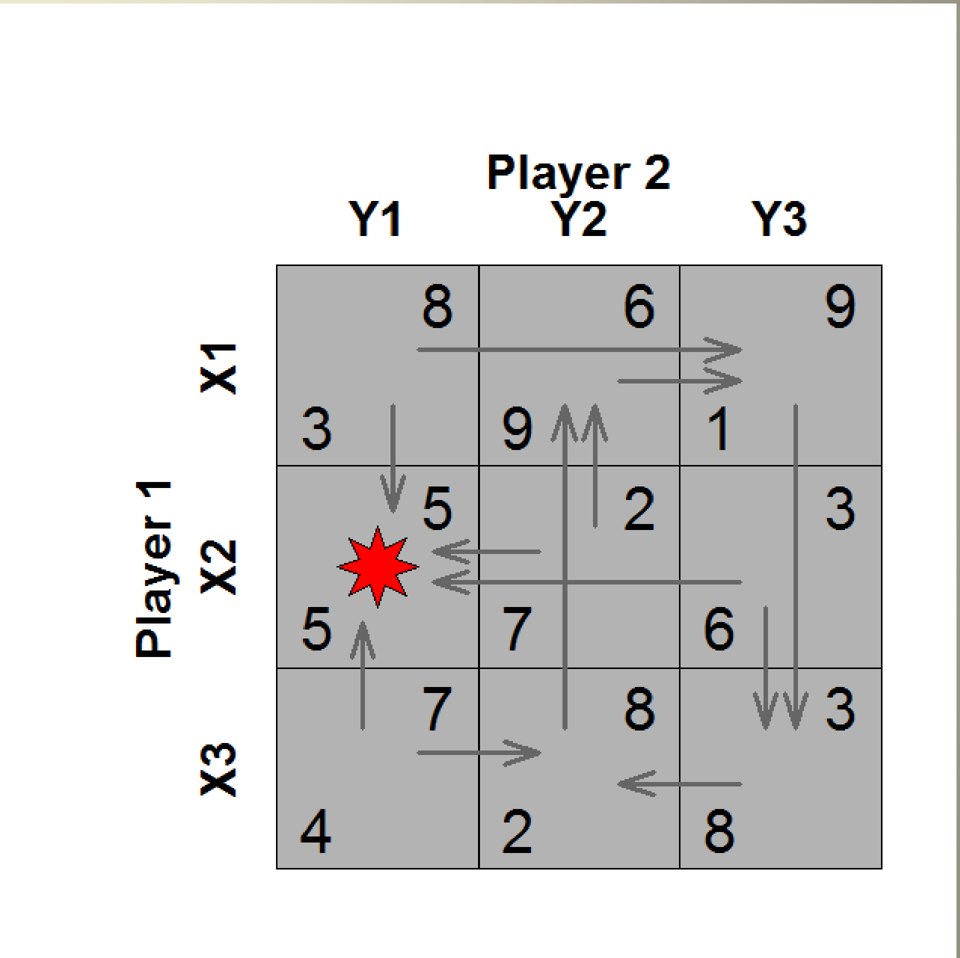
gt_brgraph (X = matrix(c(3, 0, 1, 2), 2),
Y = matrix(c(2, 0, 1, 3), 2),
P1 = "Man", P2 = "Woman", labels1 =c("Football", "Opera"),
br = TRUE)



Playing games with R

- Some examples:

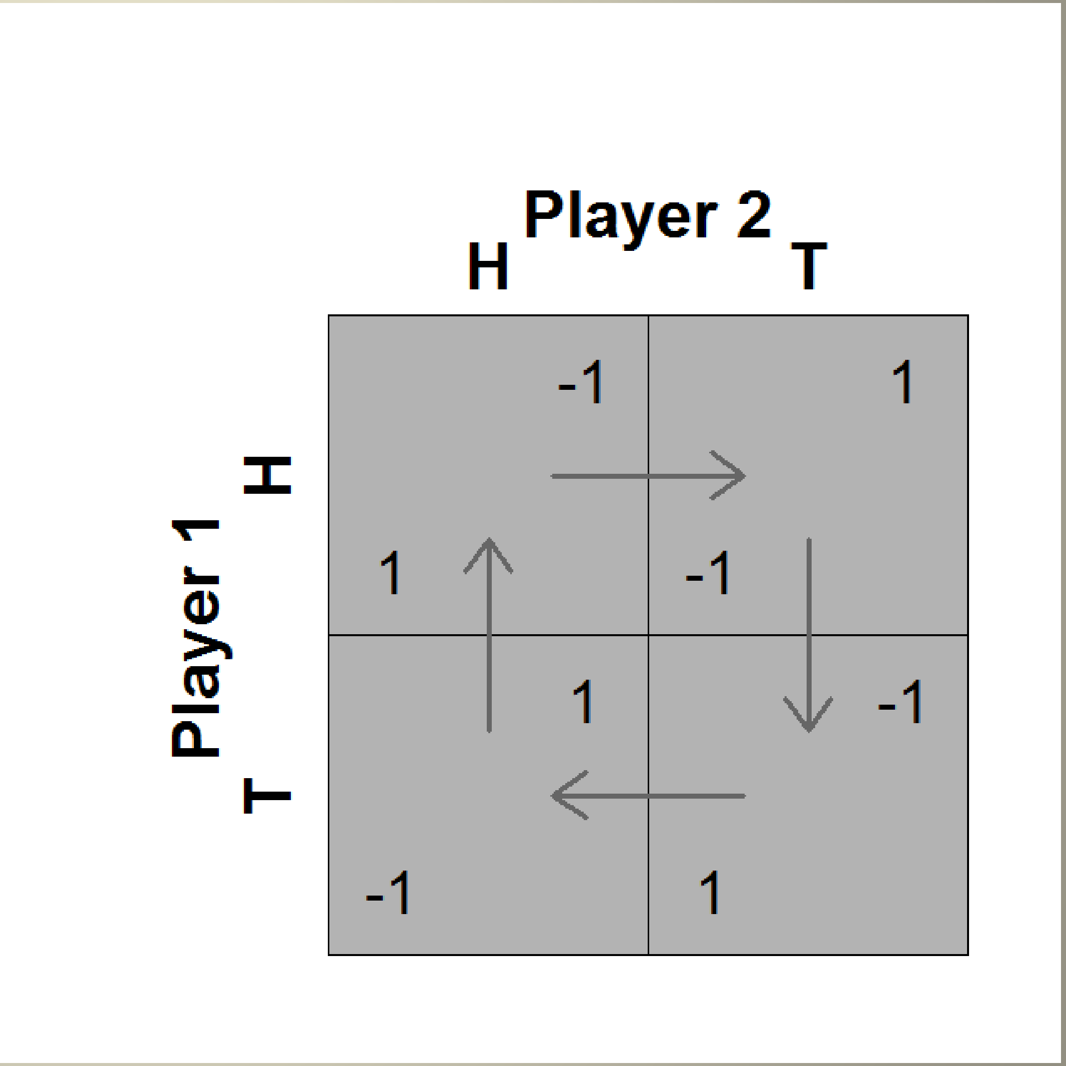
gt_bimatrix(X = matrix(c(3, 5, 4, 9, 7, 2, 1, 6, 8), 3),
Y = matrix(c(8, 5, 7, 6, 2, 8, 9, 3, 3), 3), P1 = "Player 1", P2 = "Player 2",
labels1 = NULL, labels2 = NULL)



Playing games with R

- Some examples:

```
gt_bimatrix(X = matrix(c(1, -1, -1, 1), 2),
Y = matrix(c(-1, 1, 1, -1), 2),
P1 = "Player 1",  P2 = "Player 2",
labels1 = c("H", "T"))
```



Playing games with R

- Some examples:

```
gt_brgraph (X = matrix(c(1, -1, -1, 1), 2),
Y = matrix(c(-1, 1, 1, -1), 2),
P1 = "Player 1",  P2 = "Player 2",
labels1 = c("H", "T"),  br = TRUE)
```