Game Theory

5: Dynamic games in extensive form and subgame perfect Nash equilibrium
Review of lecture four

- Coordination games
- Assurance games
- Chicken games
- Strategic moves
- Nash equilibria in mixed strategies
Dynamic games

• In **static or normal games** the players choose their strategies simultaneously (or without knowing the strategies played by the other players)

• In social interactions, however, situations are common where people **act in sequence**, one after the other

• In these cases the normal form is conveniently substituted by the **extensive form**

• The extensive form is expressed by a **game tree**
To each terminal node a n-tuple of payoffs is specified, one for each player of the game, starting from the payoff for the player that took the first move, followed by the payoff for the player that took the second move, etc.
The issue of information in GT

• Dynamic games make more clearly arise the role played by information in GT

• Which kind of information problems can we have in a game (a reprise...)?
Perfect information

• Information is **perfect** when all players can observe all actions preceding their moves
• Otherwise, you have a game of **imperfect information**
• In the preceding lectures we have considered **static games** of imperfect information
• In this lecture we introduce the analysis of **dynamic games** of perfect information
• That means that in the game tree each player, while moving, **knows the node** where the game is arrived at
• That means that during the development of the dynamic game **no move is hidden**
Complete information

• A game has **complete information** when all players know exactly the preferences of the other players.

• Does a player know the type of player she is playing with (i.e., her preference ranking)? If yes, you have a game of **complete information**. Otherwise, you have a game of incomplete information!

• In the preceding lectures we have considered **static games** of imperfect and complete information.

• In this lecture we introduce the analysis of **dynamic games** of perfect and complete information.

• In the next lectures we will relax this last assumption.
The “Decolonization” game

- **A** (colony) and **B** (controlling country)
- Status quo:
  - **B** receives 4 units from the colony as a net product and gets 2 more units as taxation
  - Making a living and paying taxes leave **A** with 0 as a profit
  - Payoffs (0,6)
- **A** claims “No taxation without representation”
Two stages’ game

• **First stage**: colony A decides whether to Revolt or to Consent

• **Second stage**: ruling country B decides whether to be Soft (S) or to be Hard (H) after having observed the decision taken by colony A
  
  – Being soft after Consent means reducing taxes to 1 unit ➔ (1,5)
  
  – Being Hard means not reducing taxes ➔ (0,6)
  
  – Being soft after Revolt leads to representation ➔ (3,3)
  
  – Being Hard after Revolt generates a war of independence ➔ (6,0) (that A will win)
Tree of the Decolonization game

- Two players
- Two stages
- Three nodes
- An agent has to make a decision at each node
- Four outcomes
- Four payoffs (one for each outcome) for each player

Remember! The first payoff always corresponds to the payoff for the player who moves first, the second payoff to the player who moves second, and so on...

A

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Strategies of Decolonization game

• Colony A moves first

⇒ it has two strategies: \{R, C\}

• Controlling country B has to choose a plan of action for every possible strategy of player A (i.e., it has 2 moves available BUT how many strategies???)

⇒ The four available strategies for B:

  \{SS, SH, HS, HH\}

• SS means “play S after R and play S after C”, etc.
Backward induction

• How to solve such type of game? **Backward induction**
• This method for game solution can be applied in general to games of **perfect information**
• It consists in “**pruning the branches of the tree**”
• It starts from the edge, because of the **expectations** that the player moving in the **penultimate stage** have on behavior of the last player, and so on approaching the root of the tree
Clearly the bad guys did not know anything about game theory otherwise they would have anticipated the final equilibrium since the beginning...or maybe it was rational after all...under which conditions?

For example, if they were unsure about Holmes’ abilities (type?
More on that later...)
"pruning the branches"

The final outcome is (3,3). But what about the equilibrium strategies that produce such outcome?
Games and subgames

- Dynamic games may be considered as composed of a sequence of stages
- This property is pointed out by the idea of subgame
- Subgame is a subset of an extensive form game satisfying the following properties
  - It begins at a node
  - It includes all nodes following that node and no others

Previous games have three subgames (including the complete game)
Backward induction and subgame perfection

• A solution by backward induction solves by construction every subgame of a game in extensive form.

• Then a solution of a game in extensive form by backward induction is a Nash equilibrium step by step of the whole game: by backward induction a Nash equilibrium is selected that is perfect in all subgames!!!

• This is called a Subgame Perfect Nash Equilibrium (SPNE)

• But how to write it? Let’s see another example
A young emerging leader Y of a party is considering whether to challenge (ch) or not (no) the authority of the older established national leader O. Having observed Y’s move, O may decide to attack the younger challenger (at) to confirm his supremacy or abstain and co-opt her (co) into the national leading group, hoping to preserve his own premiership.

Let us define the outcomes of the game:
- (ch)+(at) → party crisis
- (ch)+(co) → diarchy
- (no)+(at) → restoration
- (no)+(co) → appeasement

And let us assume the following ordering of preference:
- Old leader : restoration > appeasement > diarchy > crisis
- Young leader : appeasement > diarchy > restoration > crisis
SPNE for the game

- Numbers respect preference orderings and can help the intuition
- O chooses (co) if Y chooses (ch) and chooses (at) if Y chooses (no)
- Knowing that, Y chooses (ch) and the game ends with (2,3)
- **How you write the SPNE?**
  - (ch; co, at)
- Why writing “at” if player O will never play “at” in equilibrium?
The role of counterfactual scenarios...

https://www.youtube.com/watch?v=B6wJq9AZVfY&t=2s
SPNE for the game

- So...why writing “at” if player O will never play “at” in equilibrium?
- Because if Y would expect O to play “co” if he plays “no”, he would do that!
- It is the (rational) expectation that O would play “at” if he plays “no”, that convinces Y to play “ch” (expecting O to play “co” in that case...)
- That is, if we write (ch; co) we do not fully capture the strategic reasoning made by players! And it is not a surprise, cause in this case **we are not writing a solution in terms of strategies, but just in terms of moves!!!!**
• **SPNE:** (ch; co, at)
• Besides... is this SPNE the only NE of the game?
• The problem arises whether there are other Nash Equilibria of a game in extensive form that are **not subgame perfect**
• In such case, what kind of NE are selected by backward induction? Are there some **particular properties** belonging exclusively to SPNE (compared to NE)?
In normal form you write down the strategies!
The game has 8 outcomes, however resulting payoffs are always 4
Two NE:
• (ch)+(co-at) → (2,3) (diarchy)
• (no)+(at-at) → (1,5) (restoration)

Why does backward induction eliminate the second NE?
Non credible threats or promises

The only possibility to reach the result (1,5) in a sequence of moves is the following one:

1. O, trying to avoid Y’s challenge, threats her to play (attack) if she plays (challenge)
2. Then Y would play (no) trying to avoid her worst result and giving O the opportunity to get his best result

- But why should Y trust this threat? Once the young leader chooses (ch) the old one is rationally induced to play (co) anyway, getting 3 instead of 1
SPNE are those NE that rule out solutions implying accepting non-credible threats or promises in one or more subgames
Deterrence game

• An aggressive nation A is deciding whether to demand (y) or not (n) a concession from a near pacific nation B
• B may concede (c) or oppose (o) to A’s claim
• If B plays (o), A can decide to attack (a) or desist (d)

A’s ordering of results: (cap)>(sq)>(war)>(ret)
B’s ordering of results: (ret)>(sq)>(war)>(cap)
Solution of Deterrence game

A’s ordering of results: 
(cap)>(sq)>(war)>(ret)

B’s ordering of results: 
(ret)>(sq)>(war)>(cap)

By backward induction

• At its second move A will choose (a) as it prefers (war) to (ret)
• Knowing that, B has to choose between (war) and (cap) and will choose (o)
• Knowing that B knows that, at its first move A chooses (n) as it prefers (sq) to (war)
• The SPNE: (n,a;o)

isplaying a firm attitude may deter aggressive intentions
Solution of Deterrence game

A’s ordering of results:
(cap)>(sq)>(war)>(ret)

B’s ordering of results:
(ret)>(sq)>(war)>(cap)

SPNE: (n,a;o)

• In such equilibrium nobody (actually) moves!!!
• Note that this does not imply that we do not have a game! Even if we do not see any social interaction, the strategic interaction is there!
• There is so much social interaction in (just a seeming) inaction! And we discover it with a game!
When deterrence is useless

G’s ordering of results:
(cap) > (war) > (sq) > (ret)

P’s ordering of results:
(ret) > (sq) > (war) > (cap)

By backward induction
- At its second move G will choose (a) as it prefers (war) to (ret)
- Knowing that, P has to choose between (war) and (cap) and will move (o)
- Knowing that P knows that, at its first move G chooses (y) as it prefers (war) to (sq)

SPNE: (y,a;o)

No way to deter a “very aggressive” country
When you are “weak” and he is not

H’s ordering of results:
\((\text{cap}) > (\text{war}) > (\text{sq}) > (\text{ret})\)

U’s ordering of results:
\((\text{sq}) > (\text{cap}) > (\text{ret}) > (\text{war})\)

SPNE: \((y, a; c)\)

The Uk and France behaviour against Hitler in 1939 leading to German occupation of Czechoslovakia?
The ransom game

The story: a terrorist group has to decide if kidnapping (K) an hostage (and asking for a ransom) or not (NK). After having observed the action of the terrorist group, the home-country of the hostage has to decide how to react (paying (P) or not (NP))

Assumptions
1) If the terrorists receive the ransom, they give back the kidnapped; if they do not receive it, they kill him/her
2) The terrorist group pays a cost to kidnap someone (at least an opportunity cost)
3) The home-country of the kidnapped person pays a political cost if he/she is killed but only if it is a democratic country; if the home-country is an autocracy, no relevant political cost to pay in front of its own public opinion
The ransom game with a Democracy

With a Democracy the SPNE is (K, P)
The ransom game with an Autocracy

With an Autocracy the SPNE is (NK, NP)
The ransom game

Which kind of empirical expectation you can derive form such simple game?
✓ You expect a positive relationship between a democracy and the number of kidnapped persons (from a democracy)

About credible strategies: the only way for a Democracy to improve its pay-off is to play as an Autocracy, for example by employing a pre-commitment strategy of «no negotiations with the terrorists»

...Such strategy however must be credible: in the previous game, according to the preference ranking (i.e., payoffs) of the players, such strategy is not credible at all (and indeed is ruled out as non-credible by the SPNE)!
Exercise

How many strategies available to A? 2
And to B? $3^2=9$
(9 ways to reply with her 3 moves to the 2 moves by A)
And to C? $2^6=64$
(64 ways to reply with her 2 moves to the 6 moves by B)
Exercise

DYNAMIC GAMES IN EXTENSIVE FORM

Final outcome
(3,5,3)

SPNE: (a1; b3,b2; c1, c2,c1,c1,c2,c2)