Game Theory

5: Nash equilibrium with continuous strategies

Review of lecture four

- Coordination game and focal points
- Chicken game and strategic moves
- The concept of Mixed Strategy Nash equilibrium
- How to find MSNE in matrix games

Continuous games

- Players in some context can choose pure strategies from a **wide range** of possibilities (much larger than 2…)
- For example: philanthropists choosing charitable contribution amounts *or* parties choosing their policy location on a left-right scale
- In such cases, allowing the *discreteness* in players' strategies choice-set would require us to give each player too many distinct strategies and make the game matrix simply too large (and useless as analytical tool)!
- For these games we need a different solution technique that begins with allowing **pure strategies as continuous variables**

Price competition example

- Suppose two restaurants (A and B) in a small town
- Each place has a set menu, and both restaurants have to the set the **prices** of their respective menus
- Prices are therefore their *strategic choices* in the game of competing with each other
- Each restaurant's goal is to set price to **maximize profit** (their *payoff* in this game)
- We suppose they have to get their menus printed separately w/o knowing the other's price
- As a result: we have a game of simultaneous moves!

Price competition example

- Let’s call A’s price Pa and B’s price Pb
- In setting its price, each restaurant has to calculate the consequences for its profit
- Suppose the cost of serving each customer is 8€ for each restaurant
- Suppose also that past-experience shows that when A’s price is Pa and B’s prices is Pb, the number of their respective customers (measured in hundreds per month) are given by the equations:

$$Qa=44-2Pa+Pb;\ Qb=44-2Pb+Pa$$

Price competition example

- The key idea is that if one restaurant raises its price unilaterally by 1€, its sales will go down by 200 per month (i.e., Q changes by -2) and those of the other restaurant will go up by 100 per month (i.e., part of the customers of the first restaurant will move to the second restaurant)
- The profit (Π) for restaurant A (in hundreds of euros per week) is given by the product of the net revenue per costumer (price less cost, i.e., Pa-8) and the number of costumers served:

$$\Pi a=(Pa-8)*Qa=(Pa-8)*(44-2Pa+Pb)$$

Price competition example

- Let's make a bit of algebra:

$\Pi a=-8(44+Pb)+(60+Pb)\ Pa-2(Pa)^2$

- A sets his price Pa to maximize this payoff (for each given value of Pb): i.e., A's is looking for his best-reply! The same is true of course for B

Price competition example

- How to find the value of Pa that maximizes Πa?
- ✓ By taking the *first-derivative* of course and putting it =0!
- But how to understand if you are getting the maximum rather than the minumum of Πa?
- ✓ By taking the *second-derivative* of course!
- ✓ When the second derivative is negative, congratulations! You found the maximum!
- Let's go back to Πa now…

Price competition example

$\Pi a=-8(44+Pb)+(60+Pb)Pa-2(Pa)^2$

First derivative (i.e., $d\Pi a/dPa$) = 60+Pb-4Pa

Let’s put the first derivative=0

60+Pb-4Pa=0

Is it a maximum? Yes! The second derivative is: $d^2\Pi a/d^2Pa = -4$

Therefore, Pa=15+(1/4)Pb

- ✓ This equation tells us the rule for A’s best-response (i.e., the value of Pa that maximizes A’s profit given Pb)

Price competition example

Pa=15+(1/4)Pb

✓ So if Pb=16, best-reply for A is Pa=19

Symmetrically: Pb=15+(1/4)Pa

Therefore, which is the equilibrium in this game? As usual when both players are playing their best-reply against each other!

How to find it? First algebraically, then graphically…

Price competition example

Algebraically, we need to find the point in which the two functions intersect…

Pa=15+(1/4)Pb, that is: Pa=15+(1/4)(15+(1/4)Pa)

From here: Pa=20; symmetrically: Pb=20

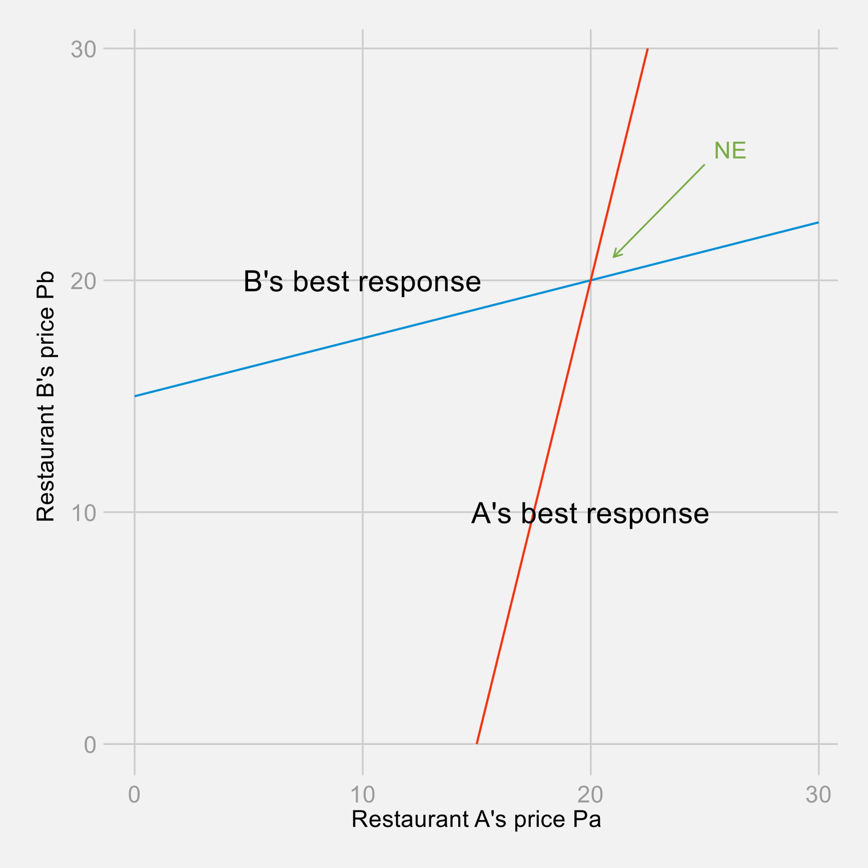
In equilibrium, each restaurant charges 20€ for its menu and makes a profit of 12€ on each of the 2,400 customers that it serves each month (i.e., Qa=44-2Pa+Pb, that is: 44-2*20+20), for a total profit of 28,800€ per month (i.e., Πa=(Pa-8)*Qa, that is: (20-8)*(44-2*20+20))

Price competition example

Geometrically: let's plot both best-reply functions!

Pa=15+(1/4)Pb and Pb=15+(1/4)Pa

If Pb=12, then Pa best-reply is to play 18. But then Pb best-reply to that would be to play 19.5. But then Pa best reply would be to play 19.875. But then…till 20/20 is played!



Political campaign advertising example

Consider an election contested by two candidates. Each is trying to win votes away from the other by advertising – either positive (about herself) or negative (about the other) adds

To keep matters simple, suppose the voters are moved solely by the ads

Even more simply, suppose the vote share of a party equals its share of the total campaign advertising that is done

Let’s call the two candidates L and R

Political campaign advertising example

Let’s suppose that when L spends x$ million on advertising and R spends y$ million, L will get a share equals to: 100*x/(x+y) of the votes and R will get 100*y/(x+y)

For example, if the both spend 10, then L will get 50% and the same is true for R

Of course, raising money to pay for these ads includes a cost (i.e., time, efforts, etc.)! Let’s suppose all these costs are proportional to the direct campaign expenditures x and y. As a result of this…

Political campaign advertising example

L's payoff is measured by its vote % minus its advertising expenditure, i.e., 100*x/(x+y)-x

Similarly candidate R's payoff is 100*y/(x+y)-y

So let's estimate the best-responses for each candidate! For a given strategy x of candidate L, candidate R chooses y to maximize its payoff, and vice-versa

We need therefore in the former case hold x fixed and setting the derivative of 100*y/(x+y)-y with respect to y equal to 0

Political campaign advertising example

How to compute the first derivative of a ratio? Let's apply the *quotient rule*

Example: $f(x) = \frac{3-2x-x^2}{x^2-1}$

Define $\mathrm{g}(x) = 3 - 2x - x^2$; $\mathrm{h}(x) = x^2 - 1$

Therefore: $f(x) = \frac{\mathrm{g}(x)}{\mathrm{h}(x)}$, then: $\frac{df}{dx} = \frac{\frac{dg}{dz}\mathrm{h}(x) - \frac{dh}{dz}\mathrm{g}(x)}{\mathrm{h}(x)^2}$

In words: (the first derivative of the numerator multiplied by the denominator *minus* the first derivative of the denominator multiplied by the denominator)/the squared of the denominator

In our case:

$$\frac{df}{dx} = \frac{(-2-2x)\left(x^2-1\ \right)-2x\left(3-2x-x^2\right)}{(x^2-1)^2}$$

Political campaign advertising example

This first derivative is equal to: $10\sqrt{x} - x - y = 0$

Minimum or maximum? Maximum of course! Given that the second derivative is negative (i.e., -1)

We can also write the first derivative for candidate R as: $y = 10\sqrt{x} - x$

For candidate L, we will have: $x = 10\sqrt{y} - y$

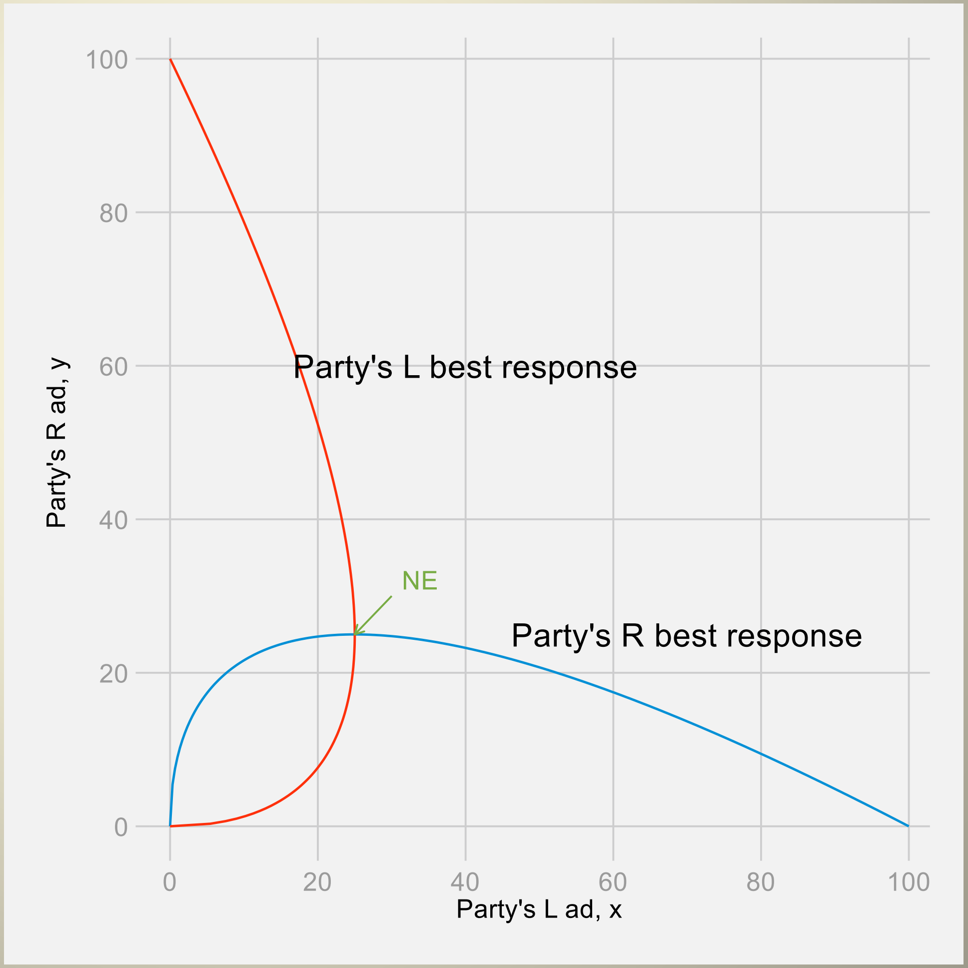
Let’s look now for the NE! When the two best-response functions intersect? Let’s begin with candidate R:

$$y = 10\sqrt{10\sqrt{y} - y} - 10\sqrt{y} + y$$

From here: y=25; symmetrically: x=25

Political campaign advertising example

Graphically:



Political campaign advertising example

We have a Prisoner Dilemma! If both candidates cut back on their ads in equal proportions, their vote shares would be entirely unaffacted, but both would save on their expenditures…and so both would have a larger payoff!

For example, if x and y=10, the payoff increases from 25 to 40 for both candidates!

However…that's not a NE!

General Method

The method for finding the NE of a game with continuous strategies as we discussed is perfectly general for any number of players

In the general approach, player 1 regards the strategies for player 2, 3…as outside his control, and chooses his own strategy to maximize his own payoff, i.e., you find an equation for player 1's optimal choice of *x* given *y*, *z*,…, that is player 1's best-response function

Similarly you can find the best-response functions for each of the other players

Home-assignment

Let's suppose that the two candidates are not symmetrically situated anymore…

Let's suppose that candidate R is able to advertise at a lower cost, because it has favored access to the media

For example, R's payoff is now: 100*y/(x+y)-y/2, while L's is the usuale one: 100*x/(x+y)-x

What would happen to the NE?