Game Theory

8 – Games of Incomplete information: dynamic games

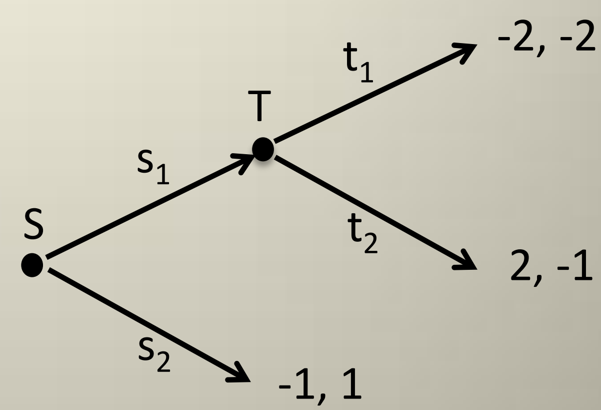
Incomplete information and estensive form

- Even in some dynamic setting, we can have different **types** but NO possibility to **update** beliefs by players according to actions undertaken during the game **whenever** the “type players” do not move FIRST!
- Let us consider an electoral college where plurality rule is in effect
- In a multi-partisan system, party T is the usual winner presenting a popular local candidate
- Considering the opportunity to interrupt this tradition, party S may choose to present one of its national leader (s_1) or to give up (s_2)
- In case (s_1) is played, T may decide to answer presenting itself a national leader (t_1) or to continue with the local candidate (t_2)

“Choosing the candidate” game (1)

For **T** the best outcome is the **status quo** (it wins with no costs)
Behaving **as usual** against the challenge is the second (the contest is uncertain but a national leader is saved for other situations)
The last result is **accepting the challenge** (the result is uncertain but a national leader is lost for other situations)

For **S** the best result is **challenging** with no reaction by T (high probability of winning)
The second best is **not to challenge** at all (it loses but the national leader is saved
The last result is when its **challenge is accepted** (no better chance to win and one national leader lost)



The solution it immediate by backward induction:

- SPNE (s_1 ; t_2)
- The game develops with a challenge ignored
- The outcome is (2,-1)

"Choosing the candidate" game (2)

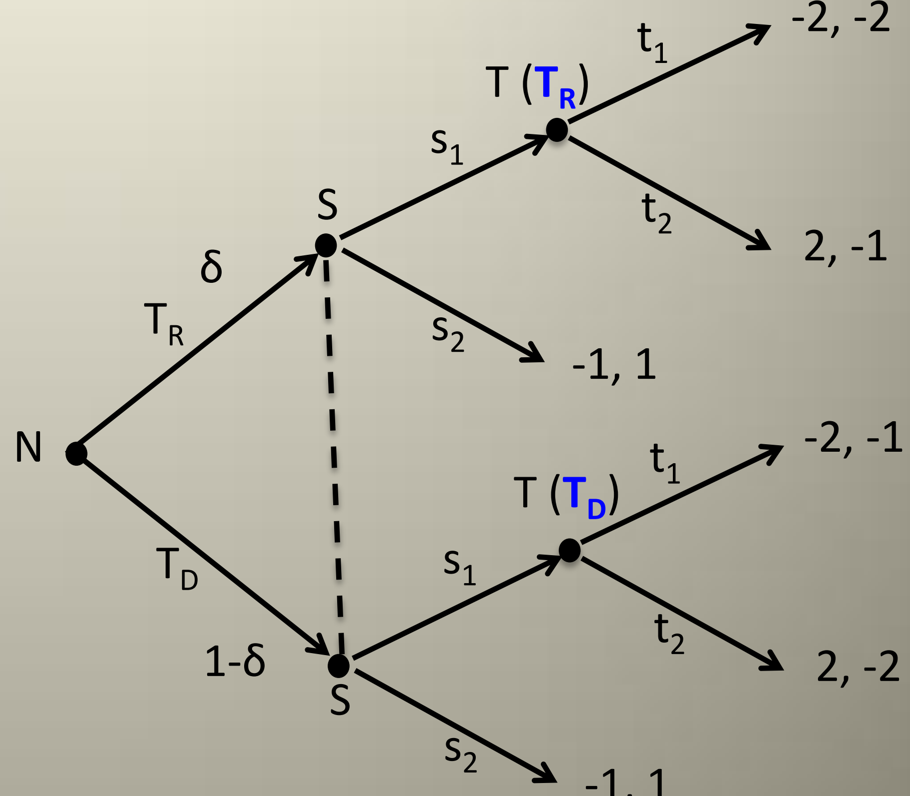
Let us **now** suppose that the challenging party S is uncertain about party T preferences

1. T is considered **preferring to lose** the college than to spend a national leader
2. Or T is considered **preferring to win** the college than to go without a leader in other competitions

In other words T may be represented by **the type T_R** (ready to loose the college) or by **the type T_D** (determined to get the seat)

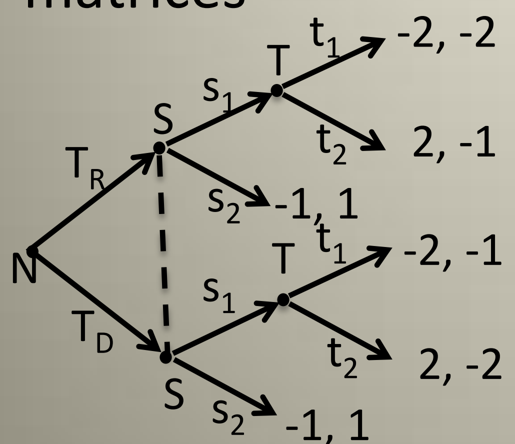
Uncertainty can be represented by an **initial move of the Nature** choosing the type T_R (with probability δ) or T_D (with probability 1- δ) of party T

N's move is private information of player T



"Choosing the candidate" game (3)

Passing to the normal form, the game is among three types of players: S, T_R and T_D. As you can see we have two proper subgames (plus the subgame of the entire game) As usual for three players games the Bayesian normal form of the entire game implies two matrices



T_R

S		t_1	t_2
	s_1	-2 , -2	2 , -1
	s_2	-1 , 1	-1 , 1

T_D

S		t_1	t_2
	s_1	-2 , -1	2 , -2
	s_2	-1 , 1	-1 , 1

There are eight strategy profiles that are:

(s_1,t_1,t_1); (s_1,t_1,t_2); (s_1,t_2,t_1); (s_1,t_2,t_2); (s_2,t_1,t_1); (s_2,t_1,t_2); (s_2,t_2,t_1); (s_2,t_2,t_2)

Note that playing t_1 for T_R or t_2 for T_D wouldn't be consistent with SPNE.

Therefore…only the two profiles **(s_1,t_2,t_1)** and **(s_2,t_2,t_1)** survive

"choosing the candidate" game (4)

In this case the probabilities δ that S assigns to T_R and $1-\delta$ that S assigns to T_D are unknown

We need to see if values exist of δ such that one or the other survived profiles **(s_1,t_2,t_1)** and **(s_2,t_2,t_1)** are not dominated for S

The expected utilities of party S for the two profiles are

T_R prob δ

S		t_1	t_2
	s_1	**-2 , -2**	**2 , -1**
	s_2	**-1 , 1**	**-1 , 1**

$$u_s(s_1,t_2,t_1) = 2\delta-2(1-\delta) = 4\delta-2$$
$$u_s(s_2,t_2,t_1) = -\delta-(1-\delta) = -1$$

T_D prob $1-\delta$

S		t_1	t_2
	s_1	**-2 , -1**	**2 , -2**
	s_2	**-1 , 1**	**-1 , 1**

➔ $u_s(s_1,t_2,t_1) > u_s(s_2,t_2,t_1)$ if and only if $\delta > ¼$

➔ The game has two BNE equilibria:

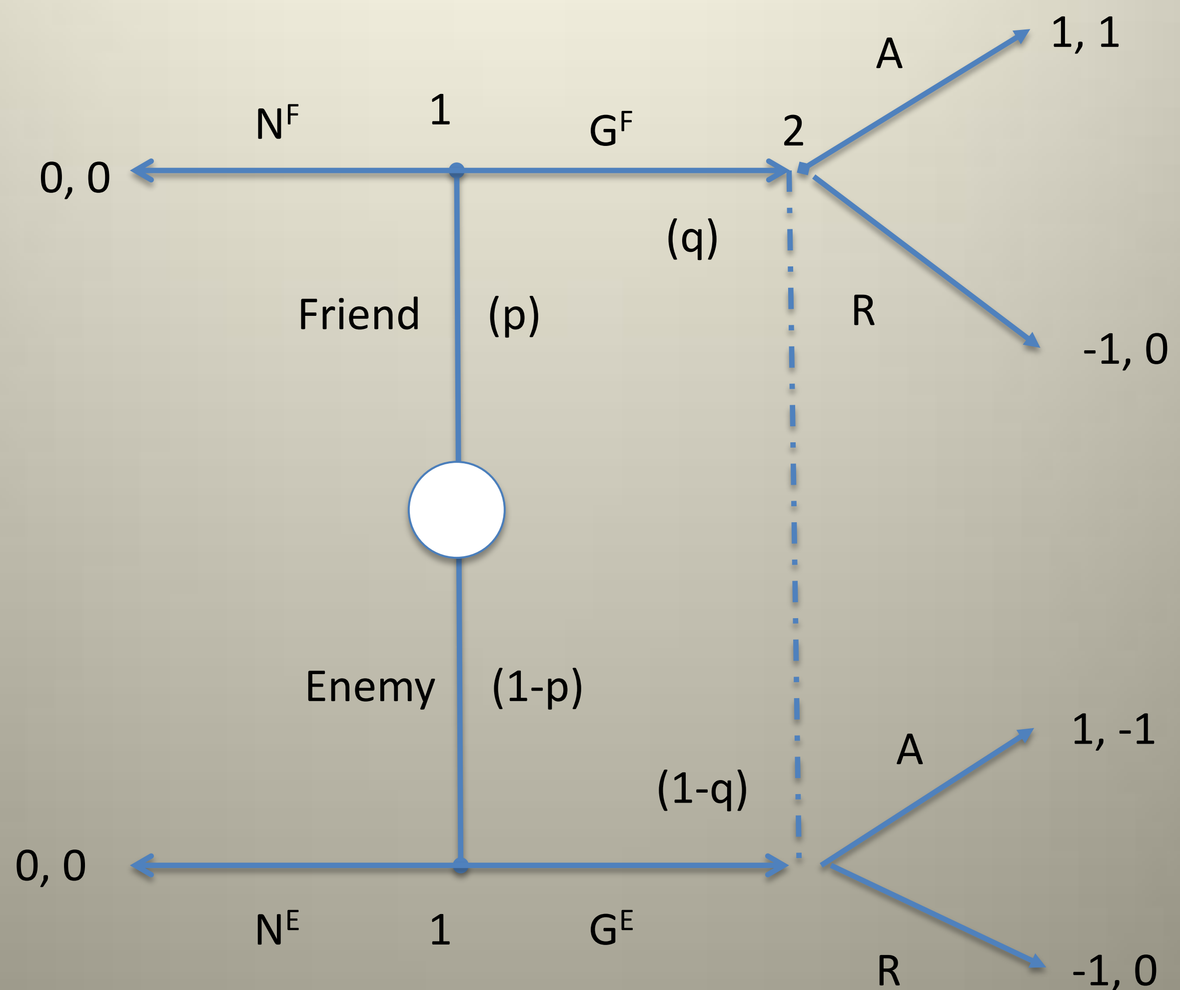
1. **$\{(s_1,t_2,t_1)$, $\delta > ¼$**} (when S believes that the incumbent party has at least 25% probability of being T_R , i.e. disposed to lose the college, S runs a national leader and T answers accordingly to its character)
2. **$\{(s_2,t_2,t_1)$, $\delta < ¼\}$** (S continues to present the local candidate and T reacts as before, if S believes that the incumbent party has at least 75% probability of being T_D , i.e. resoluted to keep the college seat)

Let’s discuss some more examples

Bayesian games with updating of beliefs

- Consider the following game: **the gift game**. *Friend* tends to keep desirable objects in his pocket to offer you as a gift, the *Enemy* no (such as rocks or frogs…). In this variant of the game, player 2 prefers to accept a gift **only** from a Friend
- Player 1 however can be of two different types: Friend or Enemy

Bayesian games with updating of beliefs



Bayesian games with updating of beliefs

- This game is a dynamic one where player 2 can update his/her beliefs about player 1 type according to what player 1 does…

Bayesian games where the updating of beliefs is possible

- There is indeed a difference, in the previous game, between **p** (i.e., the initial belief about player 1's type) and **q** (player 2's updated belief about player 1's type, **after that player 2 observes** the strategy of player 1)
- For example, suppose that player 1 behaves according to strategy N^F, G^E; thus player 2 now expects a gift from an enemy with (an updated) probability equal to 1, i.e., q=0.
- In general, player 2 has an updated belief about player 1's type, **conditional on arriving at player 2's information set** (that is, conditional on receiving a gift in our example)
- How to include such a possibility into an equilibrium?

A Perfect Bayesian Equilibrium

- PBE is a solution concept that **incorporates** *sequential rationality* and *consistency of beliefs*.
- *Sequential rationality*: players maximize their payoffs from each of their information sets (**on** or **off** the equilibrium path! More on this later…)
- How to reach that? *Consistency of beliefs*! In a PBE player's 2 updated beliefs should be **consistent** with Nature's probability distribution **and** player 1's strategy
- In general consistency between nature's probability distribution (**p** in the previous example), player 1's strategy, and player 2's updated belief (**q** in the previous example) can be evaluated by using **Bayes rule**

Bayes rule

- Bayes rule gives the **conditional probability** of an event when another event has been observed, i.e., it gives us a criterion to determine **how new information should change our beliefs about a given event**

Bayes rule

- Let p(A) and p(B) two a-priori probability of the events A and B
- Let us write **p(A|B)** the probability of the event A when B has been observed
- **Bayes rule is a formula for determining p(A|B)**
- More formally:
- $p(A|B) = (p(A)\ p(B|A))/(p(A)\ p(B|A) + p(B)\ p(B|A)$

where: p(A) is the a priori probability of A before occurring B, p(B|A) is the conditional probability of B given A, and p(B) is the a priori probability of B

Bayes rule

- **An example:**
- You are on the train and you want to understand if the person sitting next to you is a centre-right voter
- You know a priori that 54% of the Italian citizens are centre-right voters (46% centre-left)
- Now the person sitting next to you open a newspaper. You know that the 35% of centre-right voters read that newspaper (while it is read by 65% of centre-left voters)
- Which is your update belief that the person sitting next to you is a centre-right voter?
- $p(CR|N) = p(CR)\ p(N|CR)/(p(CR)\ p(N|CR)+p(CL)\ p(N|CL)) =$

$=.54*.35/(.54*.35+.46*.65) = .387$

Bayes rule

- **Bayes rule:**
- More in general, given $(h_1, h_2, \ldots, h_n)$ a set of mutually exclusive and exhaustive events compatible with the event k, then:

$$p\left(h_1 \middle| k\right) = \frac{p\left(h_1\right) p\left(k \middle| h_1\right)}{\sum_{i=1}^{n} p\left(h_i\right) p\left(k \middle| h_i\right)}$$

Bayes rule

- Going back to previous game:
- Suppose that the probability to meet a Friend in the previous example determined by Nature is ½
- Let's further suppose that the two types of Player 1 adopt the following strategy (N^E,G^F)
- Before that strategy, p(FRIEND)=1/2. Now the update probability of p(FRIEND|G^F) is…
- p(FRIEND|G^F) = (p(FRIEND) p(G^F|FRIEND))/(p(FRIEND) p(G^F|FRIEND) + p(B) p(G^F|ENEMY)…that is
- p(FRIEND|G^F) = (0.5 * 1)/(0.5 * 1 + 0.5 * 0) = 1

A Perfect Bayesian Equilibrium: definition

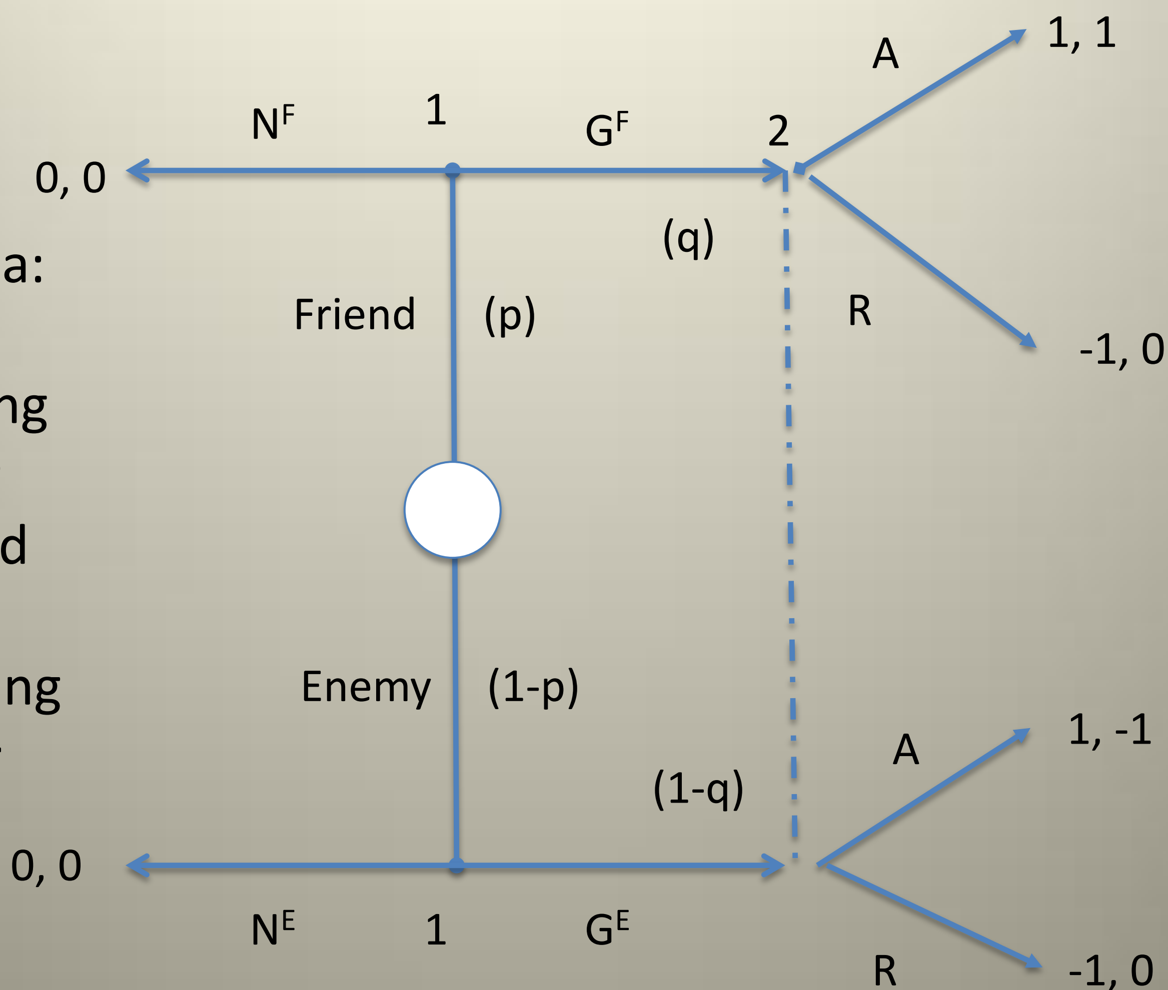
- Consider a **strategy profile** for the players (i.e., types), as well as **beliefs over the nodes at all information sets**. These are called a **PBE** if: 1) each player's strategy specifies optimal actions, **given** the strategies of the other players and her beliefs; 2) the beliefs **are consistent** with Bayes rule *wherever possible* (?!? Be patient and you will understand it…)
- In essence a PBE is a coherent story that describes beliefs and behavior in a game

A Perfect Bayesian Equilibrium: how to find it!

- Two additional terms are useful: we call an equilibrium as a **separating** one if all the types of a player behave differently
- We call an equilibrium as a **pooling** one if all the types behave the same

An application

There are **four** potential equilibria: two <u>separating</u> equilibria (featuring strategy $G^F N^E$, or strategy $N^F G^E$) and two <u>pooling</u> equilibria (featuring strategy $N^F N^E$, or strategy $G^F G^E$)



A Perfect Bayesian Equilibrium: how to find it!

- **Steps for calculating PBE:**

- Starts with a strategy for player 1 (in this case 2 strategies for the 2 types of player 1)
- If possible, calculate updated beliefs (**q** in the example) for player 2 by using Bayes rule. In the event that Bayes rule **cannot be used**, you must **arbitrarily select an updated belief**; here you will generally have to check **different potential values** for the updated belief with the next steps of the procedure;
- Given the updated beliefs, calculate player 2's optimal action
- Check whether player 1's strategy is a best response to player 2's strategy. If so, CONGRATULATIONS: you have just found a PBE!

An application

- Let's apply our procedure:
- **Separating with N^F G^E**:
- given this strategy for player 1, it be must be that $G^E|q=0$ (Bayes rule!). Thus, player 2's optimal strategy is R. But then the enemy type of player 1 would strictly prefer not to play G^E. Therefore, there is **no PBE** in which N^F G^E is played

An application

- Let's apply our procedure:
- **Separating with G^F N^E**:
- given this strategy for player 1, it be must be that $G^E|q=1$ (Bayes rule!). Thus, player 2's optimal strategy is A. But then the enemy type of player 1 would strictly prefer to play G^E rather than N^E. Therefore, there is **no PBE** in which G^F N^E is played

An application

- **Pooling with G^F G^E**:
- here Bayes rule requires that $G^E|q=p$, so player 2 optimally selects A iff p>1/2. When p>1/2 there is therefore a PBE in which q=p and (G^F G^E, A) is played - PBE: (G^F G^E, A) , q=p; p>1/2
- **Why an equilibrium?** Given the strategy (G^F G^E) played by player 1, the best reply for player 2 to that GIVEN the belief specified (q=p; p>1/2) is A. And given the strategy adopted by player 2 (A), the strategy (G^F G^E) is the best reply to that for both players!

An application

- **Pooling with G^F G^E**:
- On the other hand, in the event that p<1/2, player 2 must select R, in which case neither type of player 1 wishes to play G in the first place. Thus there is **no PBE** of this type when p<1/2.

An application

- **Pooling with N^F N^E**:
- in this case Bayes rule **does not determine** q. Why? Cause in this case both types of player 1 play N, and player 2 **cannot update** q according to Bayes rule, given that G is not played and his information set is not reached **on the equilibrium path**!!!
- Still, regardless of player 1's strategy, player 2 will have some updated belief q at his information set
- This number has **meaning** even if player 2 believes that player 1 adopts the strategy N^F, N^E. In this case, q represents player 2's belief about the type of player 1 when the "surprise" of a gift occurs (i.e., **off the equilibrium path**)

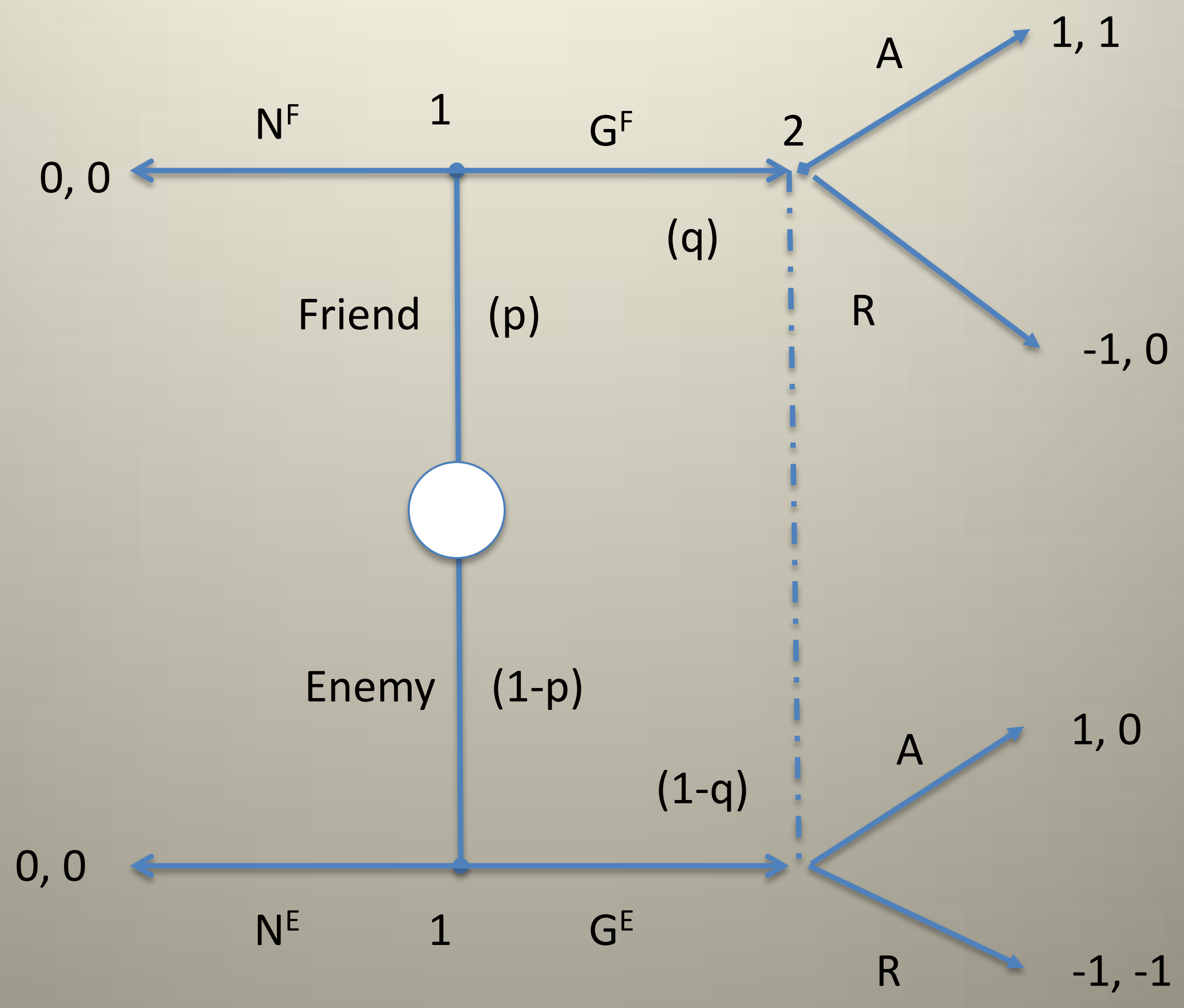
An application

- **Pooling with N^F N^E**:
- But notice that the types of player 1 prefer not giving gifts **only if player 2 selects R**
- In order for R to be chosen, player 2 must have a sufficiently **pessimistic belief** regarding the type of player 1 after the “surprise” in which a gift is given. Strategy R is optimal as long as $q<1/2$. Thus, for every $q<1/2$ there is a PBE in which player 2’s belief is q and the strategy profile (N^F N^E, R) is played
- In this equilibrium player 2 believes that an **eventual (off-the-equilibrium path) gift signals** the presence of the enemy (a misanthrope?)
- PBE: (N^F N^E, R) , $q<1/2$. On the other hand, if $q>1/2$ player 2 would select A. But then bother types of player 1 would have an incentive to switch their strategy! No PBE!

SPNE vs. PBE

- Why PBE should be considered as a refinement of a SPNE?
- Consider once again **the gift game**. However, in this variant of the game player 2 always prefer opening gifts than not opening it (*satisfying curiosity is a lovely gift by itself!*)

SPNE vs. PBE



SPNE vs. PBE

- Because the game has sequential decisions, it seems appropriate to look for SPNE. But note that the game has no proper subgame, so every NE is subgame perfect
- In particular, (N^F, N^E, R) is a SPNE (one of the two: the other is???)
- We can easily see it by analyzing the Bayesian normal form of the game. In this equilibrium, both types of player 1 choose not to give a gift and player 2 plans to refuse gifts. But this SPNE has a big problem. **Which one?**

SPNE vs. PBE

- (N^F, N^E, R) prescribes behavior for player 2 that is clearly **irrational conditional** on the game reaching his information set. Regardless of player 1’s type, player 2 prefers to accept any gift offered!
- However this preference is not incorporated into the SPNE because: 1) player 2’s information set is **not reached** on the path induced by (N^F, N^E, R), and 2) player 2’s information set does not represent the start of a subgame

SPNE vs. PBE

- Therefore, the concept of subgame perfection does not sufficiently capture **sequential rationality** (that players maximize their payoffs on or off their equilibrium path)
- On the contrary, a **Perfect Bayesian equilibrium** (PBE) does just that! The key to this equilibrium concept is that it **combines** a strategy profile with a **description of beliefs that the players have at each of their information sets**. The beliefs represent the players' assessments about each other's type, conditional on reaching different points in the game

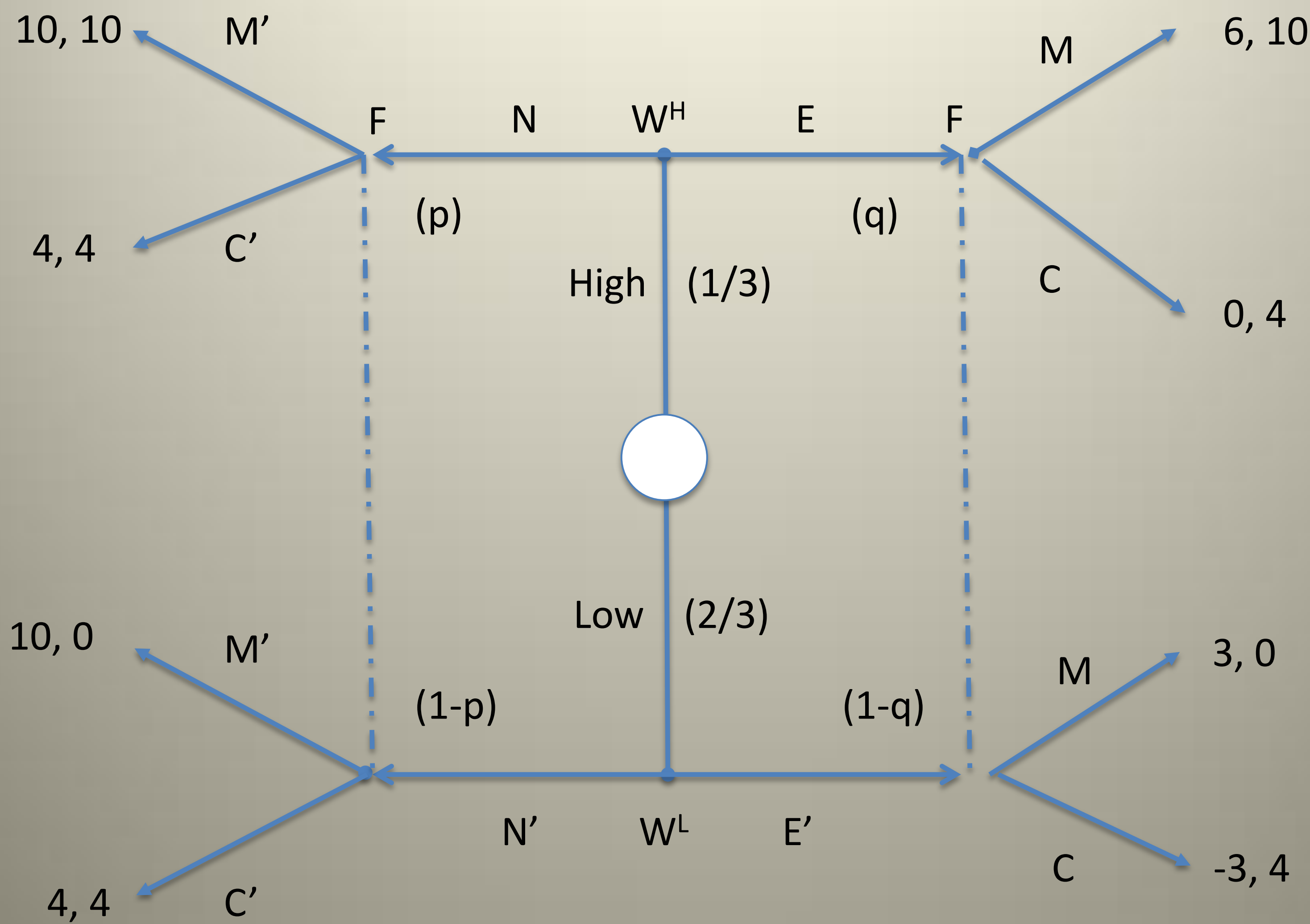
SPNE vs. PBE

- Let’s go back to the example: (N^F, N^E, R) cannot be a PBE!
- We already discussed that also in this scenario player 2 will have some **updated belief q** (not produced by Bayes rule, still…) at her information set
- Given the belief q, we can determine player 2’s optimal action at his information set. It is easy to show that action A is best for player 2 whatever is q (i.e., there is **no value of q** that could be consistent with (N^F, N^E, R) , that is, that could induce player 2 to play R)
- Thus, **sequential rationality** requires that player 2 select A, and therefore (N^F, N^E, R) is a SPNE but NOT a PBE
- So which is the PBE of this game?

Job-Marketing Signaling

- **The signaling role of education**: which role of **formal education** in the **marketplace**?
- A worker (W) and a firm (F). The worker can be of **two types**: **high** or **low type**. Firm must decides whether to employ the worker in an important managerial job (M) or in a much less important clerical job (C). M produces a benefit of 10 to both types of worker, however they have **different education costs**: the high type to get an education must pay 4 units of utility, the low type 7. C produces a benefit of 4.
- Importantly, education is of **no direct value to the firm**; the firm's payoff does not depend on whether the worker gets an education, but **only** on the intrinsic type of the worker
- The initial system of beliefs of the Firm given by Nature is: High Type=1/3; Low Type=2/3

Job-Marketing Signaling



Job-Marketing Signaling: comments

- Two PBNE: the first one is (EN’, MC’, p=0, q=1)
- **Insights:**
- **First**: the only way for the high-type worker to get the job that she deserves is **to signal her type** by getting an education. Otherwise the firm judges the worker to be a low type
- **Second**: the value of education as a signaling device depends on the types’ **differential education costs**, not on any **skill enhancement** that education deliver
- That is…to the extent that highly productive people are **more likely** than less-productive people to get degrees, than rather than helping people become smart, universities exist merely to help people who are already smart **to prove** that they are smart!

Job-Marketing Signaling: comments

- Two PBNE: the second one is (NN', CC', $p=1/3$, $q<2/5$)
- **Insights:**
- if firms are **enough pessimistic** about the chance to meet a high quality type when they observe the unexpected signal of "education" (i.e., they are quite pessimistic about the ability of the **educational costs to discriminate among types**), both workers (including the high type) will not have any incentive to get a degree

Let’s discuss some more examples

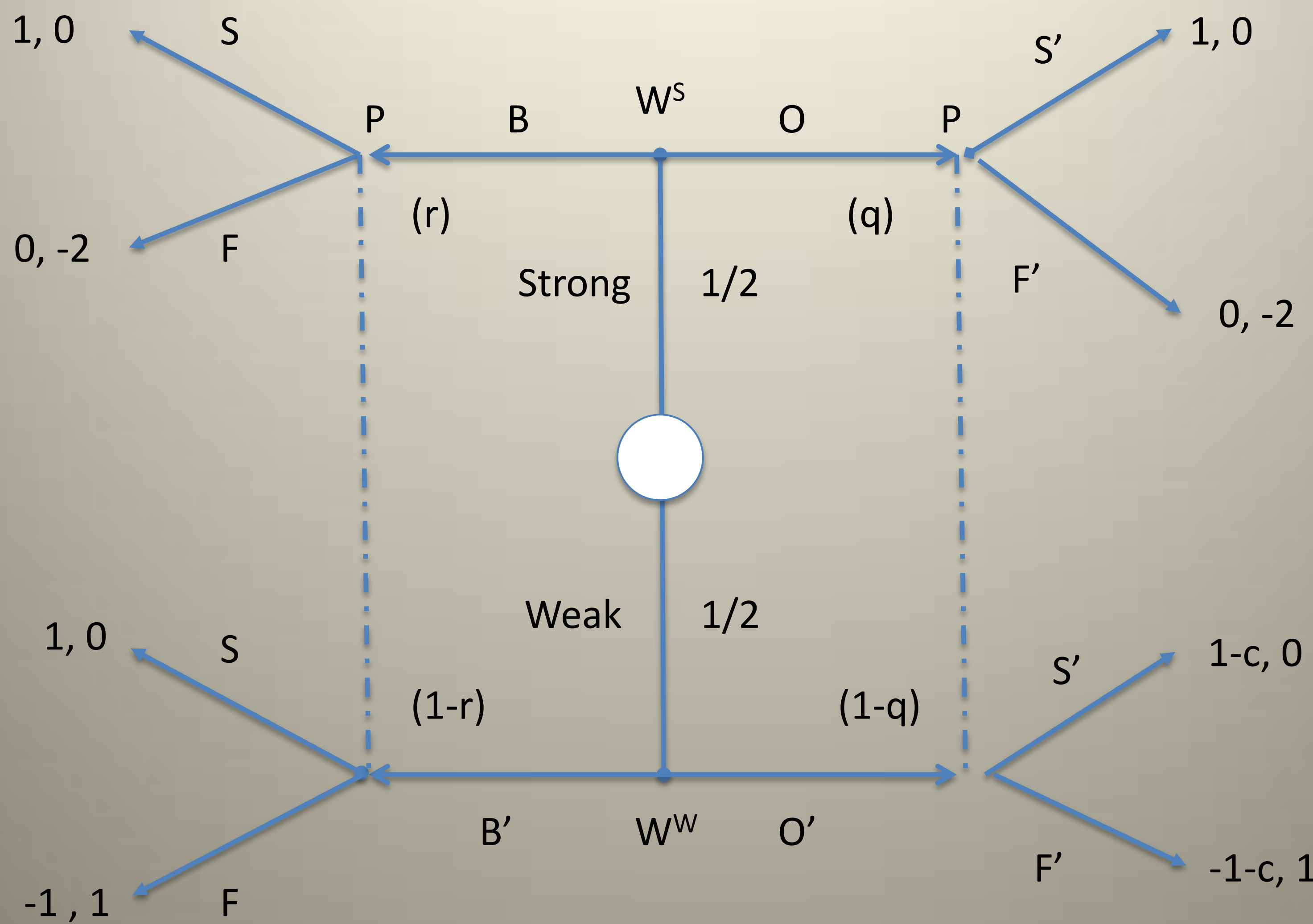
The Princess Bride

https://www.youtube.com/watch?v=njZBYfNpWoE

The Princess Bride

- Wesley (the hero) vs. Humperdinck (the evil prince)
- Two types of Wesley (weak or strong)
- Wesley is lying in a bed in the prince's castle when the prince enters the room. Wesley decides whether to get out of bed (O) or stay in bed (B). The prince decides whether to fight (F) or surrender (S) to Wesley after having seen Wesley's action
- The evil prince is an inferior swordsman (and much uglier) than Wesley, so he prefers to fight only with the weak Wesley
- The weak Wesley must pay a cost c to get out of bed
- A) what conditions on c guarantee the existence of a separating PBE?
- B) For what values of c is there a pooling equilibrium in which both types of Wesley get out of bed?

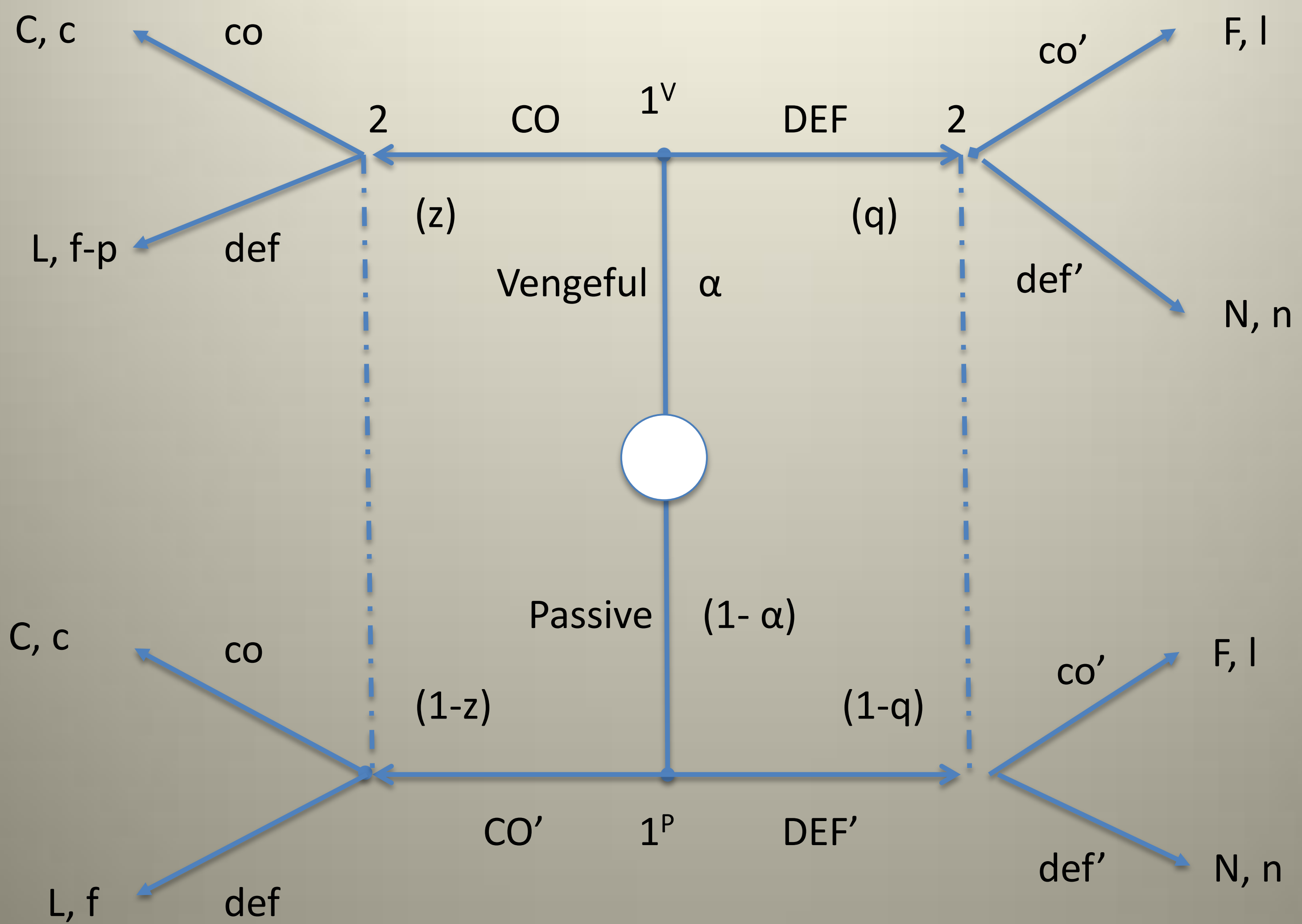
The Princess Bride



To cooperate or not to cooperate?

- There are two players called to play a dynamic version of a PD. Player 1 must decide to cooperate or not. After observing this choice, player 2 must decide to cooperate or not
- Player 1 can then decide to punish or not player 2 if (and only if) player 2 has defected after a cooperative move by player 1
- Punishing player 2 is however costly. According to Nature, we have two possible player 1: a vengeful type or a passive type. Only a vengeful type is ready to absorb the cost of the punishment (perhaps it is a benefit for him after all…), and therefore to eventually punish player 2
- The parameters of the game: F>C>N>L. The cost imposed on a “bad” player 2 is equal to P>0

To cooperate or not to cooperate?



F>C>N>L; P>0

Zhuge Liang and the Empty City

https://www.youtube.com/watch?v=sOkD68oTvHE

The story (from Sun Tzu's *The Art of War*)

- General Zhuge Liang (228 AD) had to defend the city of Xicheng from impending attack by a much larger and more powerful army.
- The general, famous among his contemporaries for his *careful strategy*, knew that he would surely face defeat in battle.
- In response, Zhuge Liang ordered his men to open the gates to the city and to remain out of sight. He then went up into a watchtower on the city walls from which, in view of anyone approaching the city, he began composing music on his zither

The story (from Sun Tzu's *The Art of War*)

Parameters

c = the value of capturing the city without a fight
$w > 0$
$y > 0$
$y > c$ (given that losing the city and the war (i.e., $-y$) is worst than losing just the city (i.e., $-c$)