Game Theory

5: Dynamic games in extensive form and subgame perfect Nash equilibrium

Review of lecture four

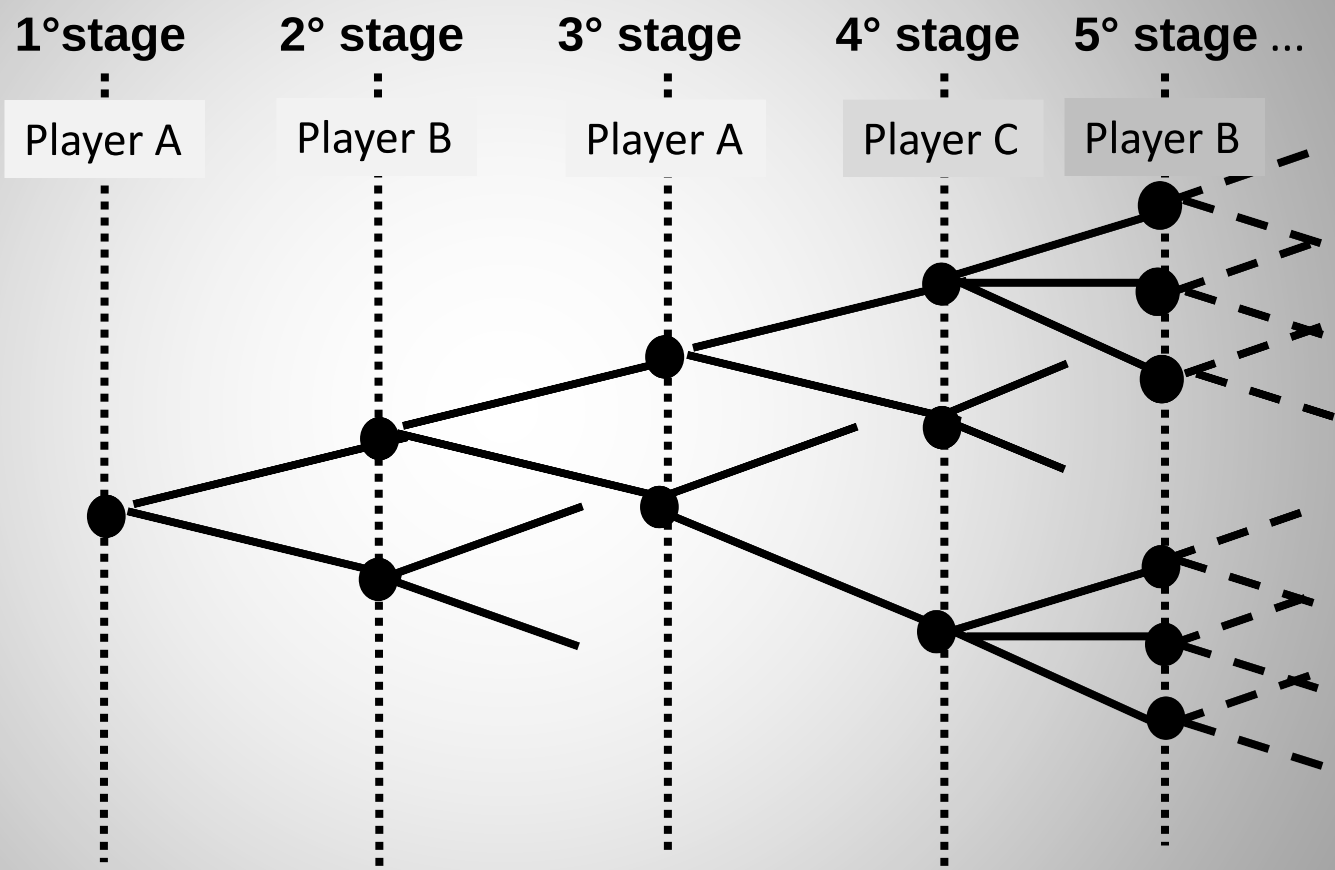
- Coordination games
- Assurance games
- Chicken games
- Strategic moves
- Nash equilibria in mixed strategies

Dynamic games

- In **static or normal games** the players choose their strategies simultaneously (or without knowing the strategies played by the other players)
- In social interactions, however, situations are common where people **act in sequence**, one after the other
- In these cases the normal form is conveniently substituted by the **extensive form**
- The extensive form is expressed by a **game tree**

Game tree example

Root
Nodes
Branches
Outcomes



To each terminal node a **n-tuple of payoffs** is specified, one for each player of the game, starting from the payoff for the player that took the first move, followed by the payoff for the player that took the second move, etc.

The issue of information in GT

- Dynamic games make more clearly arise the role played by **information** in GT
- **Which kind of information problems can we have in a game (a reprise…)?**

Perfect information

- Information is **perfect** when all players can observe all actions preceding their moves
- Otherwise, you have a game of **imperfect information**
- In the preceding lectures we have considered **static games** of imperfect information
- In this lecture we introduce the analysis of **dynamic games** of perfect information
- That means that in the game tree each player, while moving, **knows the node** where the game is arrived at
- That means that during the development of the dynamic game **no move is hidden**

Complete information

- A game has **complete information** when all players know exactly the preferences of the other players
- Does a player know the type of player she is playing with (i.e., her preference ranking)? If yes, you have a game of **complete information**. Otherwise, you have a game of incomplete information!
- In the preceding lectures we have considered **static games** of imperfect and complete information
- In this lecture we introduce the analysis of **dynamic games** of perfect and complete information
- In the next lectures we will relax this last assumption

The “Decolonization” game

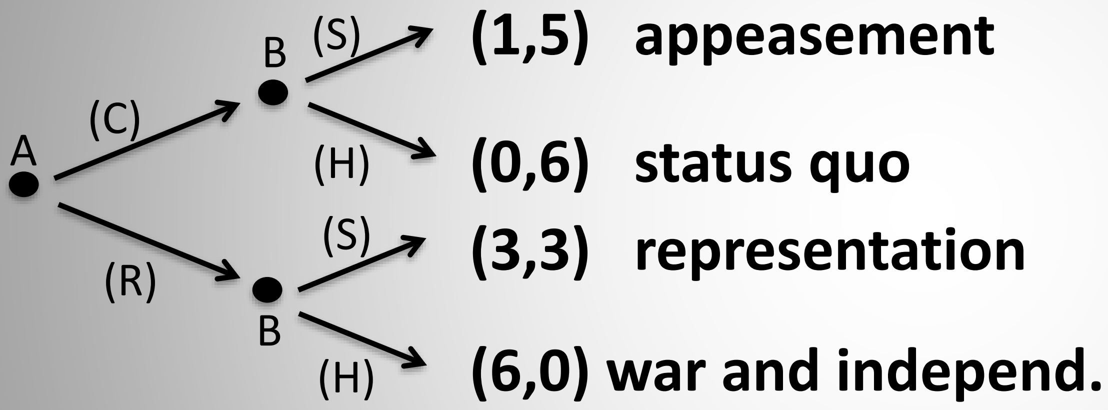
- **A** (colony) and **B** (controlling country)
- Status quo:
  - **B** receives 4 units from the colony as a net product and gets 2 more units as taxation
  - Making a living and paying taxes leave **A** with 0 as a profit

  ➔ payoffs (0,6)
- **A** claims “No taxation without representation”

Two stages' game

- **First stage**: colony A decides whether to Revolt or to Consent
- **Second stage**: **ruling country** B decides whether to be Soft (S) or to be Hard (H) after having observed the decision taken by colony A
  - Being soft after Consent means reducing taxes to 1 unit ➔ (1,5)
  - Being Hard means not reducing taxes ➔ (0,6)
  - Being soft after Revolt leads to representation ➔ (3,3)
  - Being Hard after Revolt generates a war of independence ➔ (6,0) (that A will win)

Tree of the Decolonization game



- **Two** players
- **Two** stages
- **Three** nodes
- An agent has to make a decision at **each node**
- **Four** outcomes
- **Four** payoffs (one for each outcome) for each player

Remember! The first payoff **always corresponds** to the payoff for the player who moves first, the second payoff to the player who moves second, and so on…

What is common knowledge in this game?

- Given the representation of this game, it is **common knowledge**:

1) rationality
2) strategies
3) payoffs
4) the fact that it is a game of perfect information

Strategies of Decolonization game

- Colony A moves first

➔ it has two strategies: {**R**,**C**}

- Controlling country B has to choose a plan of action for every possible strategy of player A (i.e., it has 2 moves available BUT how many strategies???)

➔What about B?

Strategies of Decolonization game

➔The four available strategies for B:

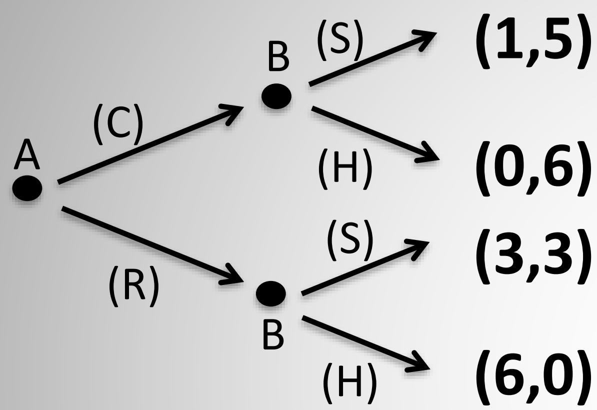
{**SS, SH, HS, HH**}

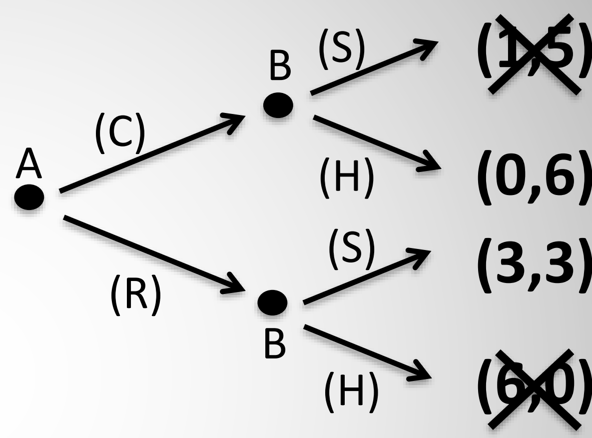
- SS means “play S after R and play S after C”, etc.
- An easy way to understand why **4 strategies**: in an extensive game, a strategy is a set of instructions **at each node**
- Player B is involved in 2 nodes, each involving 2 possible moves….from here the 4 strategies ((2 moves)^2 nodes=4)

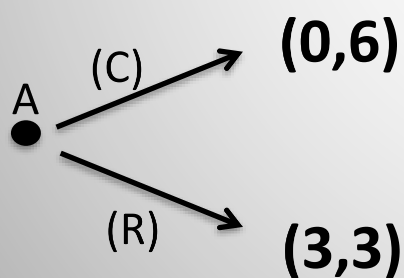
Backward induction

- How to solve such type of game? **Backward induction**
- This method for game solution can be applied in general to games of **perfect information**
- It consists in "*pruning the branches of the tree*"
- It starts from the edge, because of the **expectations** that the player moving in the *penultimate stage* have on behavior of the last player, and so on approaching the root of the tree

“pruning the branches”





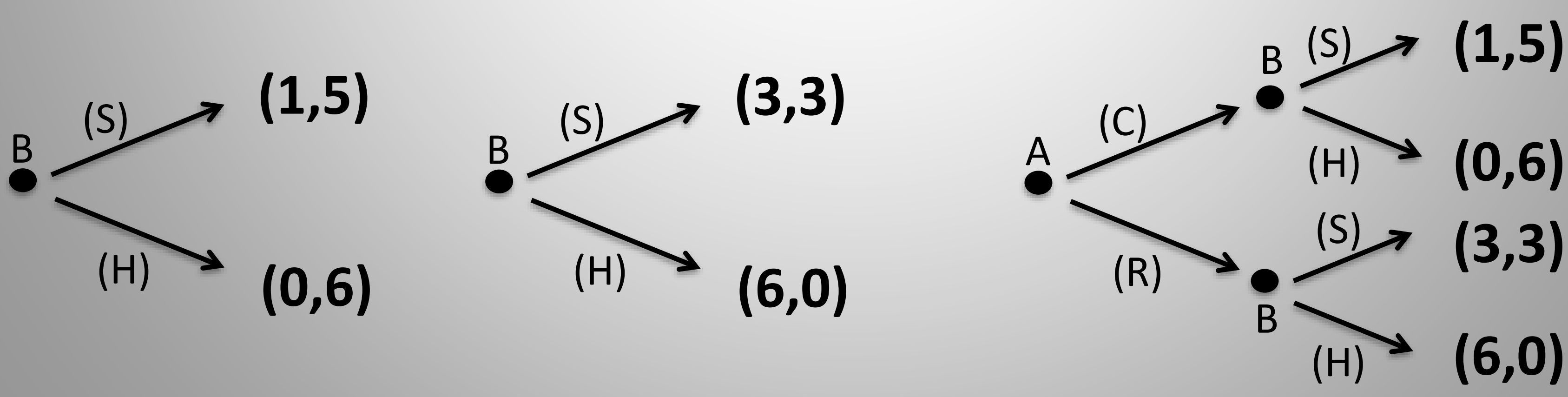


The final outcome is (3,3). But what about the **equilibrium strategies** that produce such outcome?

Games and subgames

- Dynamic games may be considered as composed of a **sequence of stages**
- This property is pointed out by the idea of **subgame**
- Subgame is a **subset** of an extensive form game satisfying the following properties
  - It begins at a node
  - It includes all nodes following that node and no others

Previous games have **three** subgames (including the complete game)



Games and subgames

- Subgames are self-contained extensive forms, **meaningful trees on their own**
- Subgames that start from nodes other than the initial node are called **proper subgames**

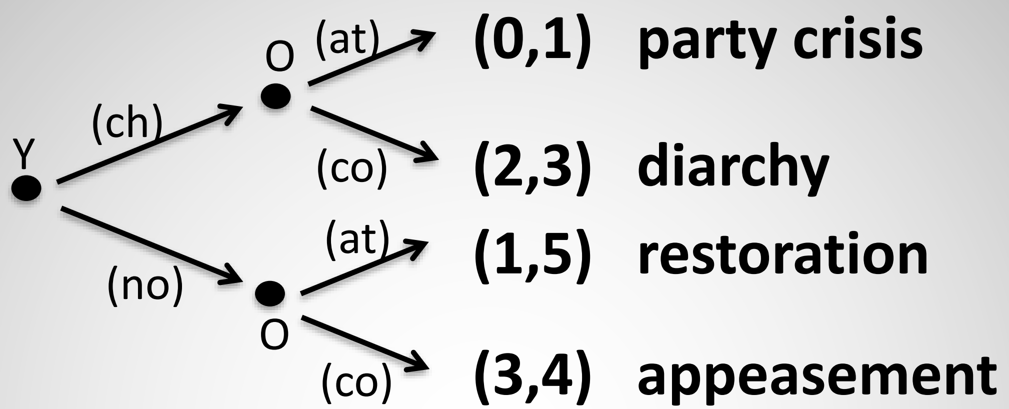
Backward induction and subgame perfection

- A solution by backward induction **solves by construction every subgame** of a game in extensive form
- Then a solution of a game in extensive form by backward induction is a **Nash equilibrium step by step** of the whole game: by backward induction a Nash equilibrium is selected that is **perfect** in all subgames!!!
- This is called a **Subgame Perfect Nash Equilibrium (SPNE)**
- But how to write it? Let's see another example

«Challenge within a party» game

- A young emerging leader Y of a party is considering whether to challenge (ch) or not (no) the authority of the older established national leader O
- Having observed Y's move, O may decide to attack the younger challenger (at) to confirm his supremacy or abstain and co-opt her (co) into the national leading group, hoping to preserve his own premiership
- Let us define the outcomes of the game:
  - **(ch)+(at) $\rightarrow$ party crisis**
  - **(ch)+(co) $\rightarrow$ diarchy**
  - **(no)+(at) $\rightarrow$ restoration**
  - **(no)+(co) $\rightarrow$ appeasement**
- And let us assume the following ordering of preference
- Old leader : restoration > appeasement > diarchy > crisis
- Young leader : appeasement > diarchy > restoration > crisis

SPNE for the game

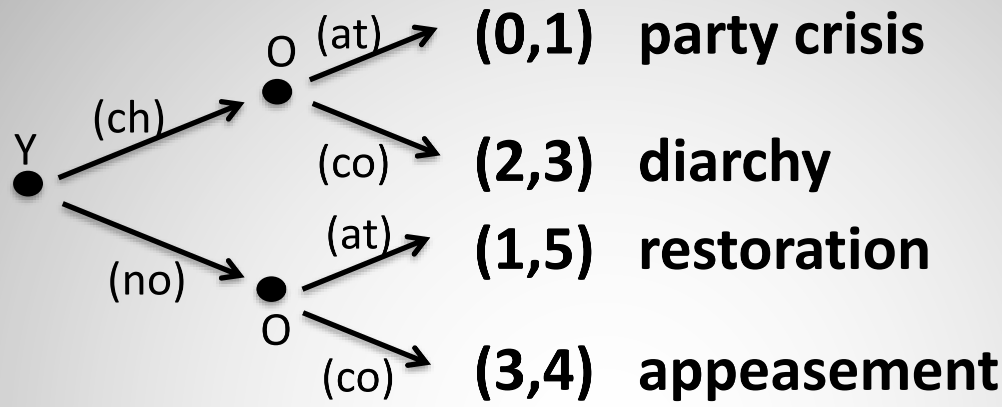


- Numbers respect preference orderings and can help the intuition
- O chooses (co) if Y chooses (ch) and chooses (at) if Y chooses (no)
- Knowing that, Y chooses (ch) and the game ends with (2,3)
- **How you write the SPNE?**

➢ (ch; co, at)

- Why writing “at” if player O will never play “at” in equilibrium?

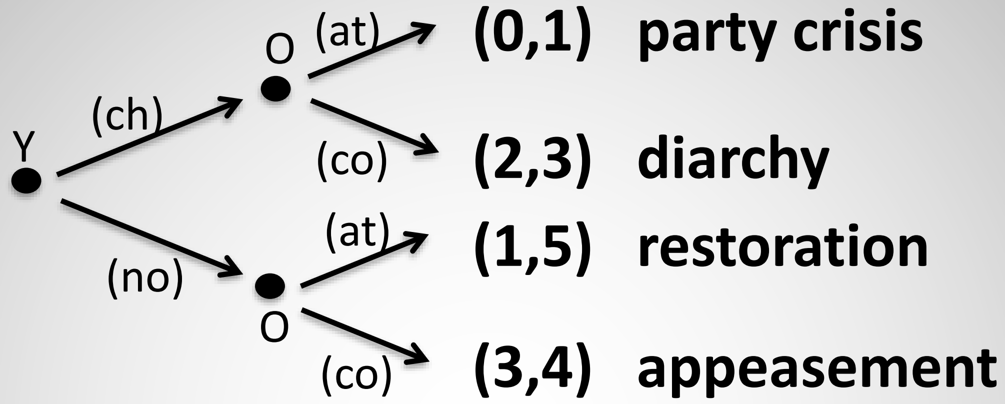
SPNE for the game



SPNE: (ch; co, at)

- So…why writing “at” if player O will never play “at” in equilibrium?
- Because if Y would expect O to play “co” if he plays “no”, he would do that!
- It is the (rational) expectation that O would play “at” if he plays “no”, that convinces Y to play “ch” (expecting O to play “co” in that case…)
- That is, if we write (ch; co) we do not fully capture the strategic reasoning made by players! And it is not a surprise, cause in this case **we are not writing a solution in terms of strategies, but just in terms of moves**!!!!

SPNE for the game



- **SPNE:** (ch; co, at)
- Besides… is this SPNE the only NE of the game?
- The problem arises whether there are other Nash Equilibria of a game in extensive form that are **not subgame perfect**
- In such case, what kind of NE are selected by backward induction? Are there some **particular properties** belonging exclusively to SPNE (compared to NE)?

«Challenge» game in normal form

		O			
		(at-at)	**(at-co)**	**(co-at)**	**(co-co)**
Y	**(ch)**	(0,1)	(0,1)	(2,3)	(2,3)
	(no)	(1,5)	(3,4)	(1,5)	(3,4)

In normal form you write down the strategies!

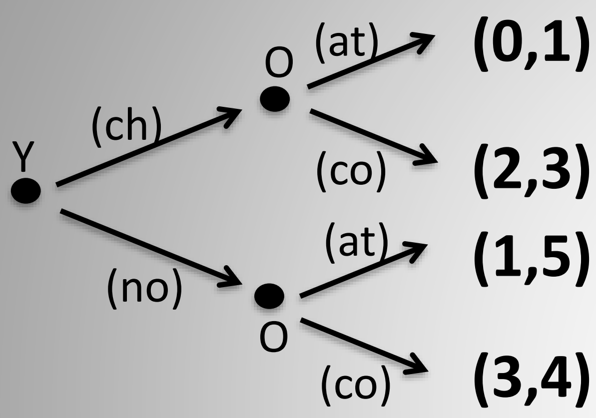
The game has 8 outcomes, however resulting payoffs are always 4

Two NE :

- (ch)+(co-at) $\rightarrow$ (2,3) (diarchy)
- (no)+(at-at) $\rightarrow$ (1,5) (restoration)

Why does backward induction eliminate the second NE?

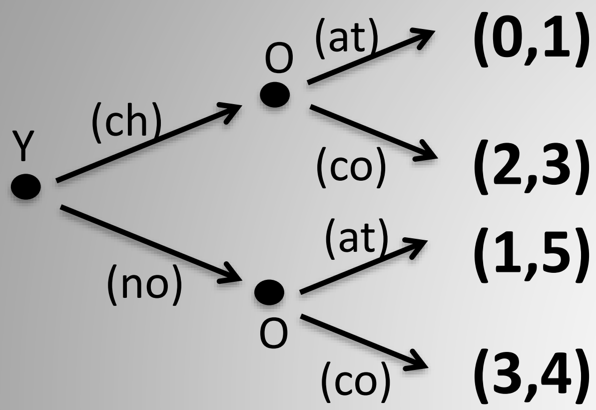
Non credible threats or promises



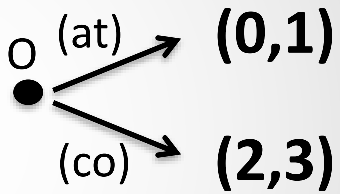
The only possibility to reach the result (1,5) in a sequence of moves is the following one:

1. O, trying to **avoid** Y’s challenge, **threats** her to play (attack) if she plays (challenge)
2. Then Y would play (no) trying to avoid her worst result and giving O the opportunity to get his best result

- But why should Y **trust** this threat? Once the young leader chooses (ch) the old one is rationally induced to play (co) anyway, getting 3 instead of 1



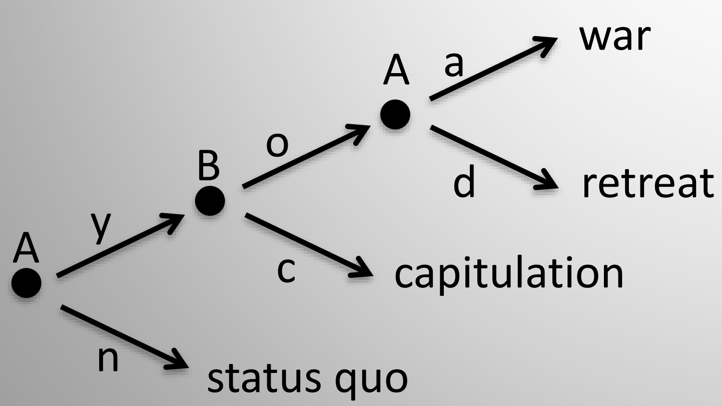
- The only way to reach the NE leading to (1,5) is **not to play a NE in one subgame**



➔ **SPNE are those NE that rule out solutions implying accepting non-credible threats or promises in one or more subgames**

Deterrence game

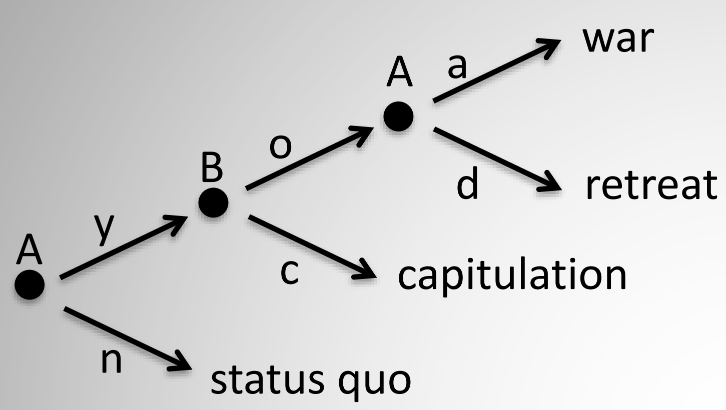
- An aggressive nation A is deciding whether to demand (y) or not (n) a concession from a near pacific nation B
- B may concede (c) or oppose (o) to A's claim
- If B plays (o), A can decide to attack (a) or desist (d)



A's ordering of results:
(cap)>(sq)>(war)>(ret)

B's ordering of results:
(ret)>(sq)>(war)>(cap)

Solution of Deterrence game



A's ordering of results:
(cap)>(sq)>(war)>(ret)

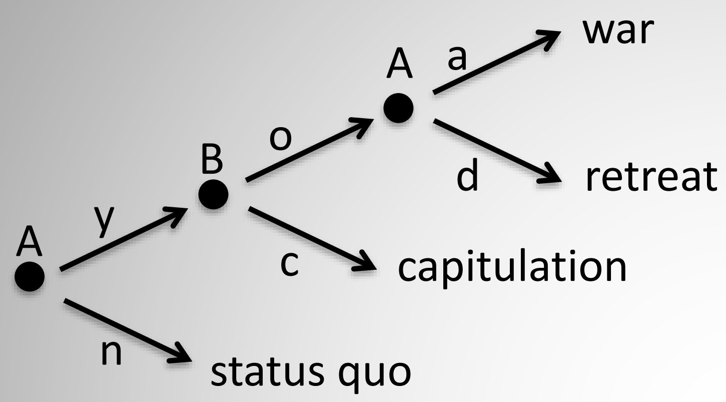
B's ordering of results:
(ret)>(sq)>(war)>(cap)

By backward induction

- At its second move A will choose (a) as it prefers (war) to (ret)
- Knowing that, B has to choose between (war) and (cap) and will choose (o)
- Knowing that B knows that, at its first move A chooses (n) as it prefers (sq) to (war)
- **The SPNE: (n,a;o)**

→ Displaying a firm attitude may deter aggressive intentions

Solution of Deterrence game



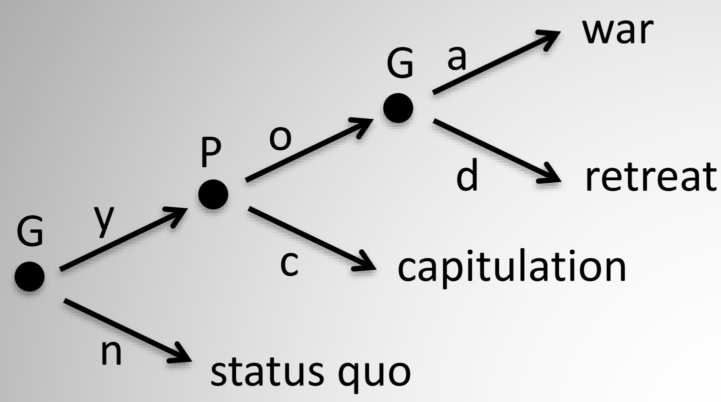
A's ordering of results:
(cap)>(sq)>(war)>(ret)

B's ordering of results:
(ret)>(sq)>(war)>(cap)

SPNE: (n,a;o)

- In such equilibrium **nobody (actually) moves**!!!
- Note that this does not imply that we do not have a game! Even if we do not see any social interaction, the **strategic interaction** is there!
- There is so much social interaction **in (just a seeming) inaction**! And we discover it with a game!

When deterrence is useless



G's ordering of results:
(cap)>(war)>(sq)>(ret)

P's ordering of results:
(ret)>(sq)>(war)>(cap)

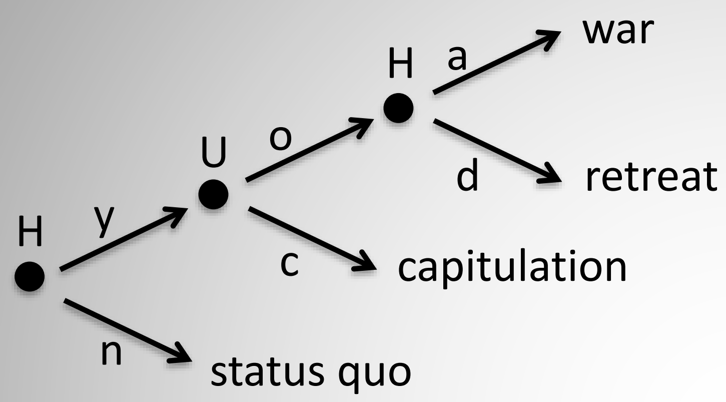
By backward induction

- At its second move G will choose (a) as it prefers (war) to (ret)
- Knowing that, P has to choose between (war) and (cap) and will move (o)
- Knowing that P knows that, at its first move G chooses (y) as it prefers (war) to (sq) ➔ (y) ➔ (o) ➔ (a) ➔ war

SPNE: (y,a;o)

No way to deter a "very aggressive" country

When you are “weak” and he is not



H’s ordering of results:
(cap)>(war)>(sq)>(ret)

U’s ordering of results:
(sq)>(cap)>(ret)>(war)

SPNE: (y,a;c)
The Uk and France behaviour against Hitler in 1939 leading to German occupation of Czechoslovakia?

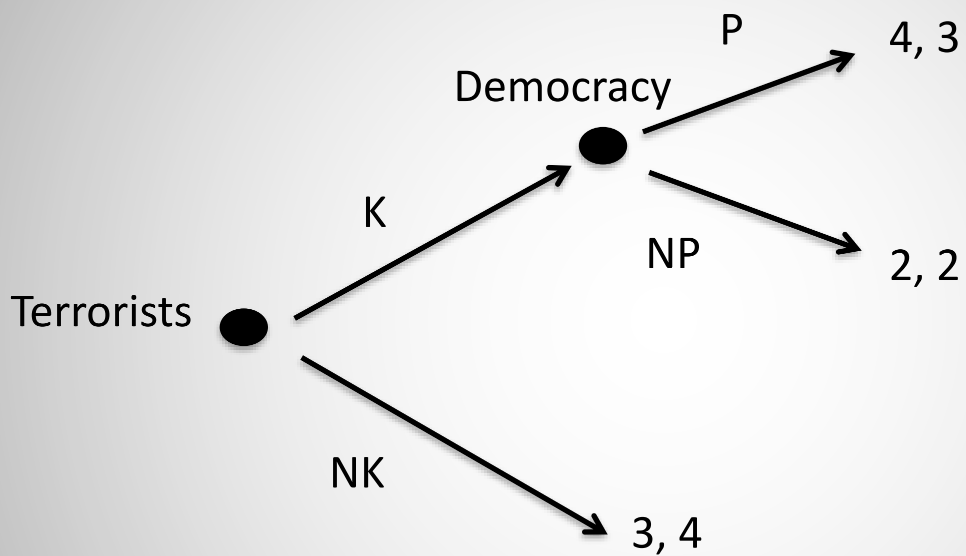
The ransom game

The story: a terrorist group has to decide if kidnapping (K) an hostage (and asking for a ransom) or not (NK). After having observed the action of the terrorist group, the home-country of the hostage has to decide how to react (paying (P) or not (NP))

Assumptions

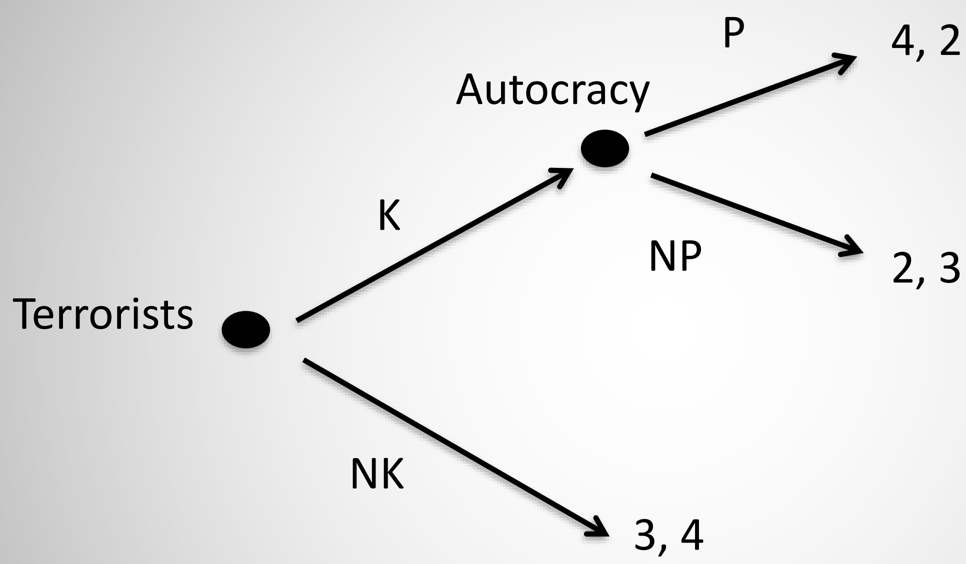
1) If the terrorists receive the ransom, they give back the kidnapped; if they do not receive it, they kill him/her
2) The terrorist group pays a cost to kidnap someone (at least an opportunity cost)
3) The home-country of the kidnapped person pays a political cost if he/she is killed but only if it is a democratic country; if the home-country is an **autocracy**, no relevant political cost to pay in front of its own public opinion

The ransom game with a Democracy



With a Democracy the SPNE is (K, P)

The ransom game with an Autocracy



With an Autocracy the SPNE is (NK, NP)

The ransom game

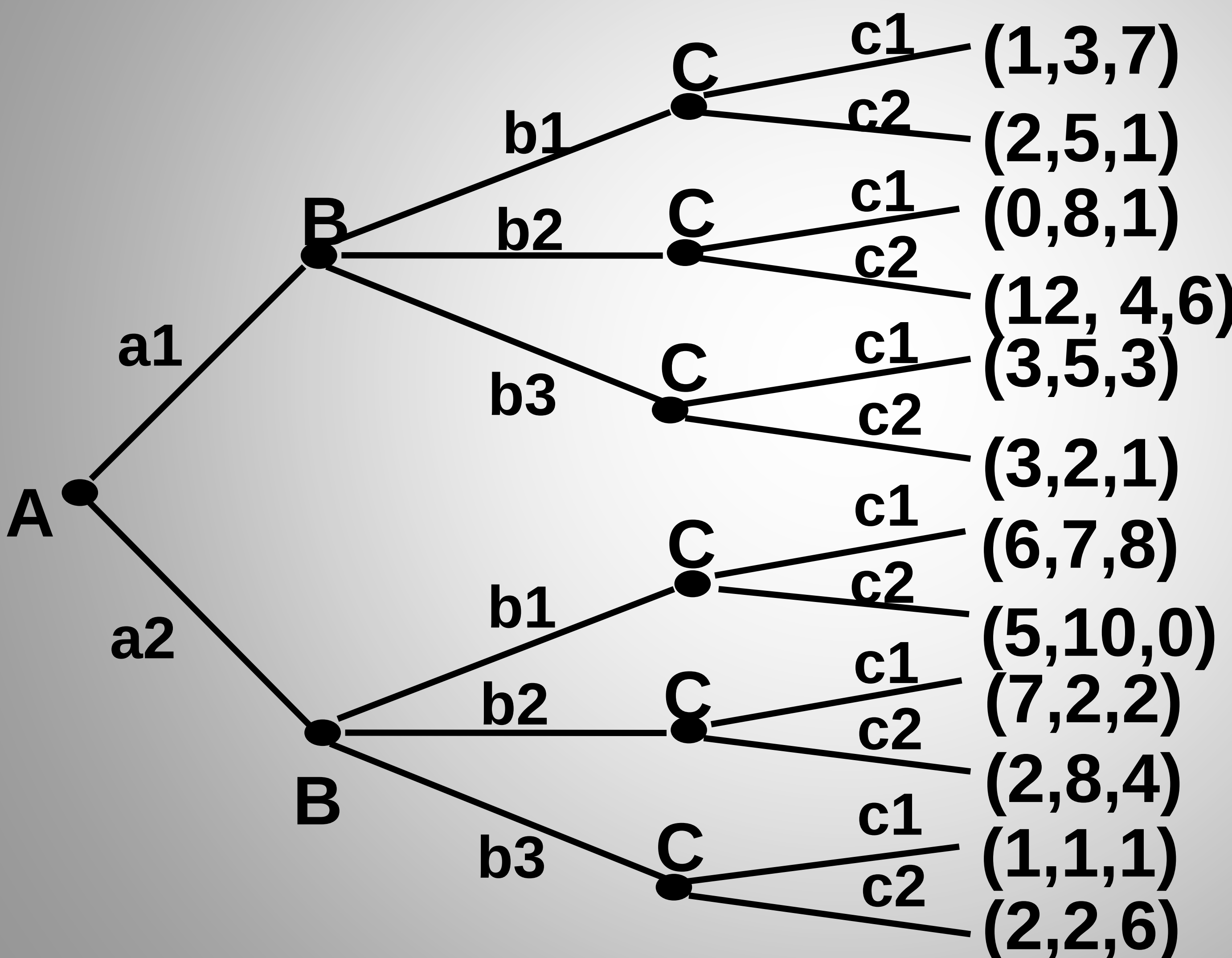
Which kind of empirical expectation you can derive form such simple game?

- ✓ You expect a positive relationship between a democracy and the number of kidnapped persons (from a democracy)

About credible strategies: the only way for a Democracy to improve its pay-off is to play as an Autocracy, for example by employing a pre-commitment strategy of «*no negotiations with the terrorists*»

…Such strategy however must be credible: in the previous game, according to the preference ranking (i.e., payoffs) of the players, such strategy is **not** credible at all (and indeed is ruled out as non-credible by the SPNE)!

Exercise



How many strategies available to the players? And find the SPNE!