Game Theory

6 – Dynamic Games with imperfect information

Review of lecture five

- Game tree and strategies
- Dynamic games of perfect information
- Games and subgames
- Backward induction
- Subgame perfect Nash equilibrium
- Dynamic games in normal form

Perfect information

- Dynamic games are games where players **move in sequence**
- If all players **know** at each stage of the game the entire development of the game before the current move, dynamic games are **games of perfect information**

Imperfect information in dynamic games

Three analytically equivalent definitions involving the rules of the game:

1. At some stage of the game some players **do not know** its entire development before the current move
2. At some stage players may move **simultaneously**
3. Some players at some stage may move while **not being observed** by the others

Static games and imperfect information

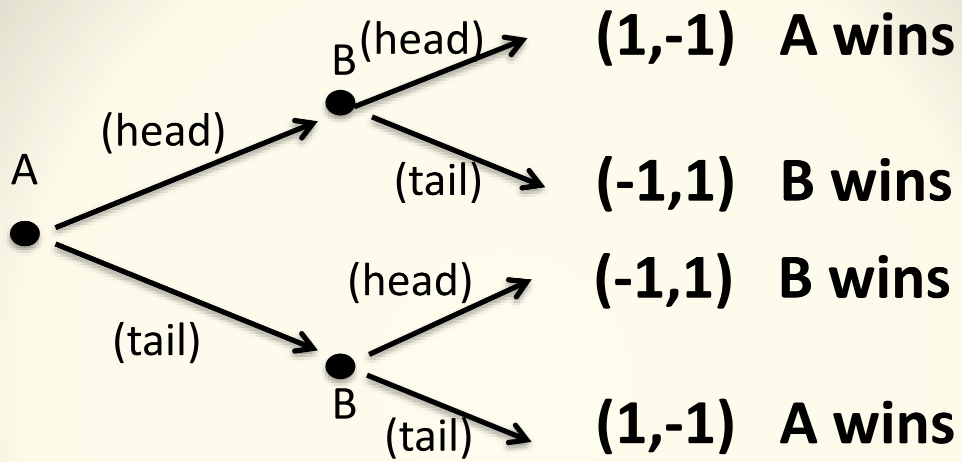
- All games in **normal form** considered so far (static games) are games of imperfect information
- When put them in extensive form it is necessary to indicate the **simultaneity** of players’ moves on the game tree
- That means that some players **do not know** from which node of the game tree they are making the move

Matching pennies in extensive form

Two players: A and B own a coin each, turned secretly on head or tail
Confronting coins, if both show the same face A takes both; otherwise B takes both

		B	
		head	tail
A	head	(1 , −1)	(−1 , 1)
	tail	(−1 , 1)	(1 , −1)

If we try to put this game in extensive form

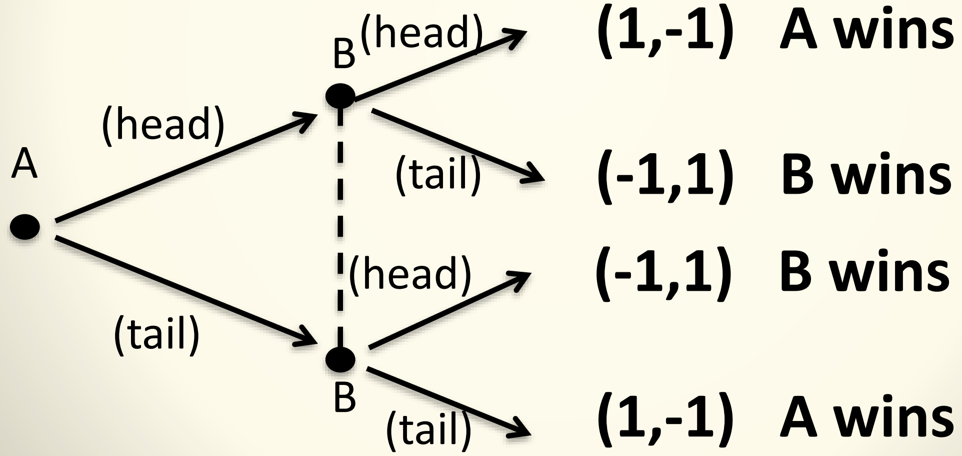


the game becomes a **stupid one**, where Bert can look at the preceding Ann’s move, turn his coin at the opposite face, and win systematically

We need to indicate on the game tree that A and B move simultaneously

A graphical convention

In games represented in extensive form (game tree) the nodes are **connected with a dashed line** if the moving player does not know from which node of the game tree he is making the move (maybe because A has moved secretly, etc.)



This game tree is **equivalent** to the previous table of the normal form

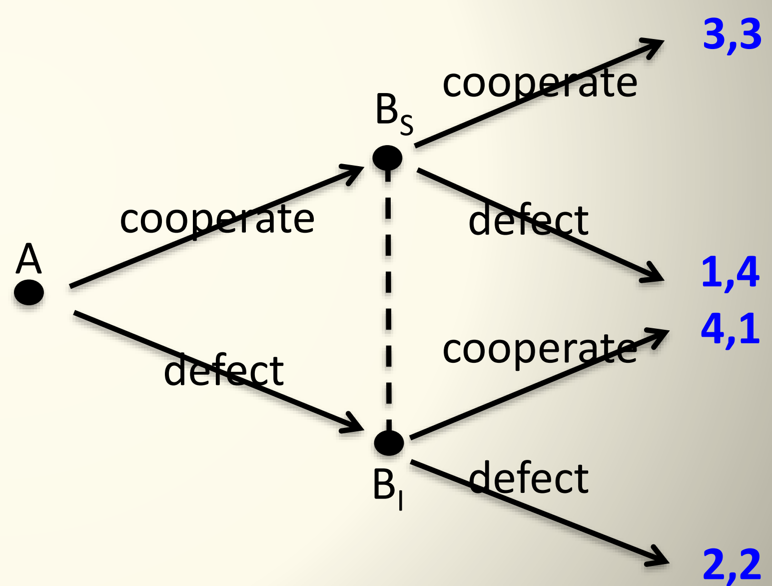
Information set

- The set of nodes that a player, when moving at a given stage, cannot exclude he is moving from, is the **information set** of that player at that stage of the game
- When information sets are all **singleton** (made of a single node) the game is one of **perfect information**
- In games of imperfect information some information sets includes **more than one node**

Prisoner dilemma in extensive form

- The just discussed graphical convention allows us to express simultaneous (static) games in extensive form
- This is the correspondence for the PD game

		B	
		cooperate	defect
A	cooperate	**(3,3)**	**(1, 4)**
	defect	**(4,1)**	**(2,2)**



- The game is symmetric: rows and columns can be exchanged in normal form
- First and second move can be exchanged in extensive form

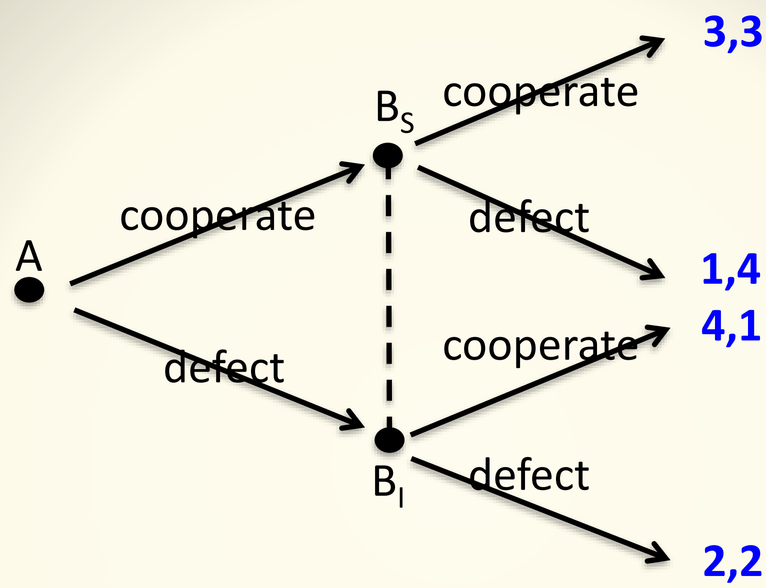
Imperfect information and backward induction

In games of imperfect information **backward induction** procedure has to be considered carefully

⇒ When a player does not know which node he is moving from he **cannot single out** the best action

⇒ The preceding player into the game tree **cannot anticipate** what he will do, **unless** one special case applies…

A special case: PD



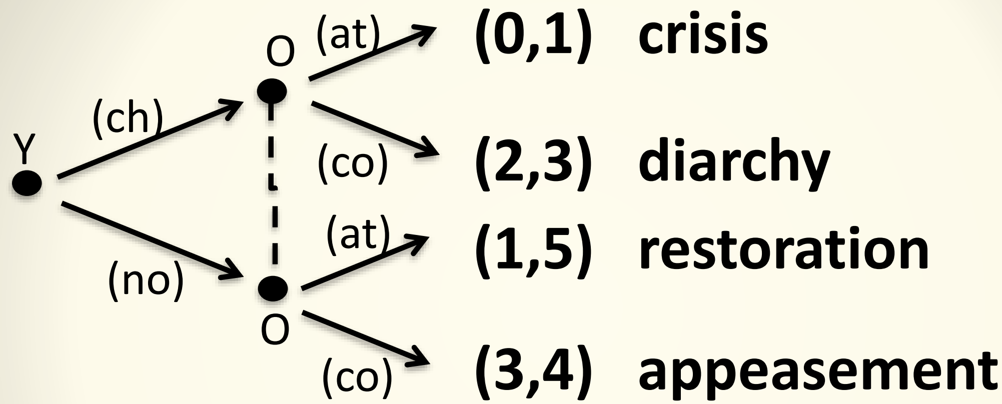
Even if B does not know if he is in B_S or in B_I he can observe that, whatever his starting node, “defect” gives him more utility then “cooperate”:

- 4 instead of 3 if A has chosen “cooperate”
- 2 instead of 1 if A has chosen “defect”

This is because “cooperate” is a dominated strategy. And this is anticipated by backward induction by player A…

The general case

- Let’s go back to the “Challenge into the party” game



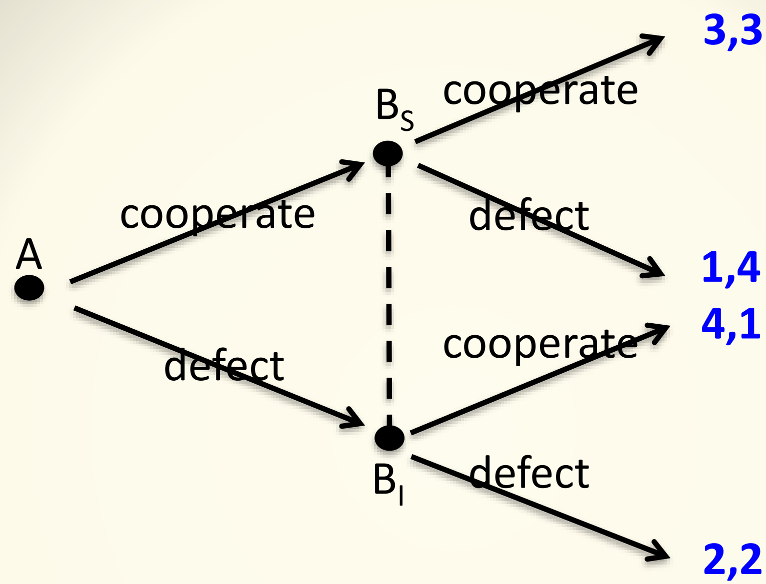
- If Y chooses “challenge”, O will choose “co-opt” as he prefers sharing the power (diarchy) to a party fission (crisis)
- If Y chooses “no challenge”, O will choose “attack” as he prefers regain his absolute power (restoration) to an agreement with his younger competitor (appeasement)
- Y **cannot anticipate** O choice and backward induction reasoning cannot start

Subgames with imperfect information

A **subgame** is a subset of the extensive form that satisfies the following criteria:

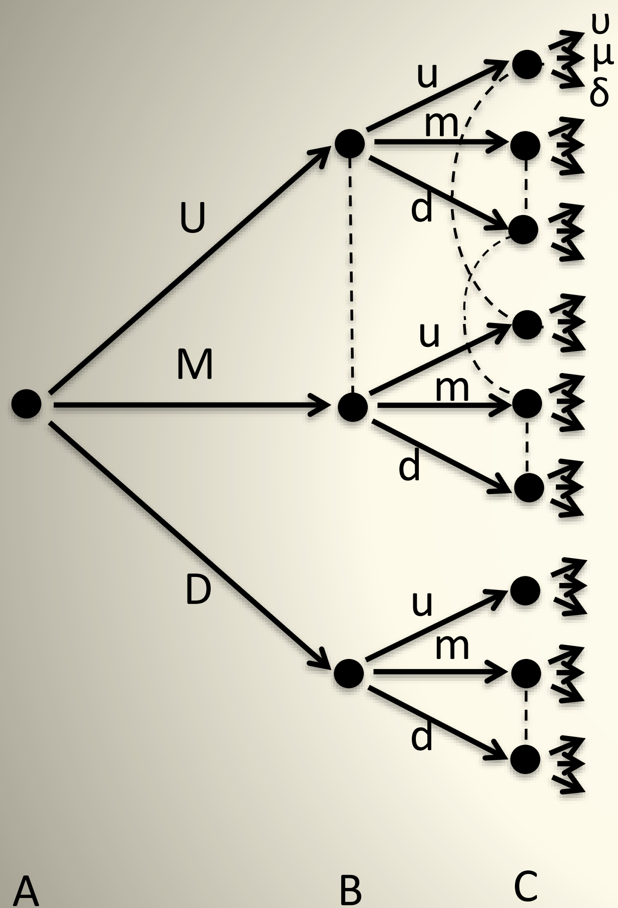
1. It begins at a node (singleton) (but we knew it already…)
2. It includes all nodes following this initial node and no others (but we knew it already…)
3. It **does not cut any information sets**: if two nodes are part of the same information set they belong to the same subgame (this is a new property!!!)

Information sets and subgames (1)



- Player A has one information set that is a singleton
- Player B has one information set with two nodes (none of which is a singleton)
- No proper subgames exist for PD **but** the whole game

Information sets and subgames (2)

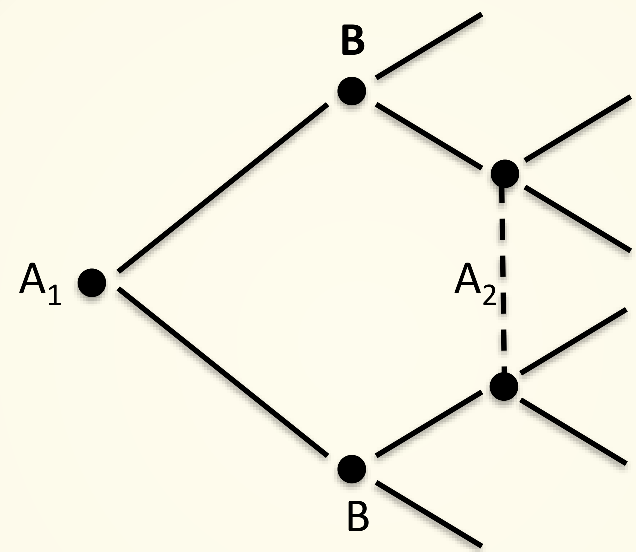


Each player A, B, C has three moves: {U, M, D} ; {u, m, d} ; {υ, μ, δ}

- At the first stage **A has one information set** (singleton)
- At the second stage B knows that A either chooses {D} or {U,M} that are the **two information sets of B**
- At the third stage C knows that B has chosen among {Dd,Dm} , {Du}, {Md,Mm,Ud,Um}, {Mu,Uu} that are **four C's information sets**

The game has **three subgames (two proper subgames)**: the original game, a (proper) subgame following D, a (proper) subgame following the path D-u

A conventionally excluded information set



This game is admissible only if player A in A_2 has forgotten what she has done in A_1

We assume that all players remember their past actions

Games are interactions with **perfect recall**

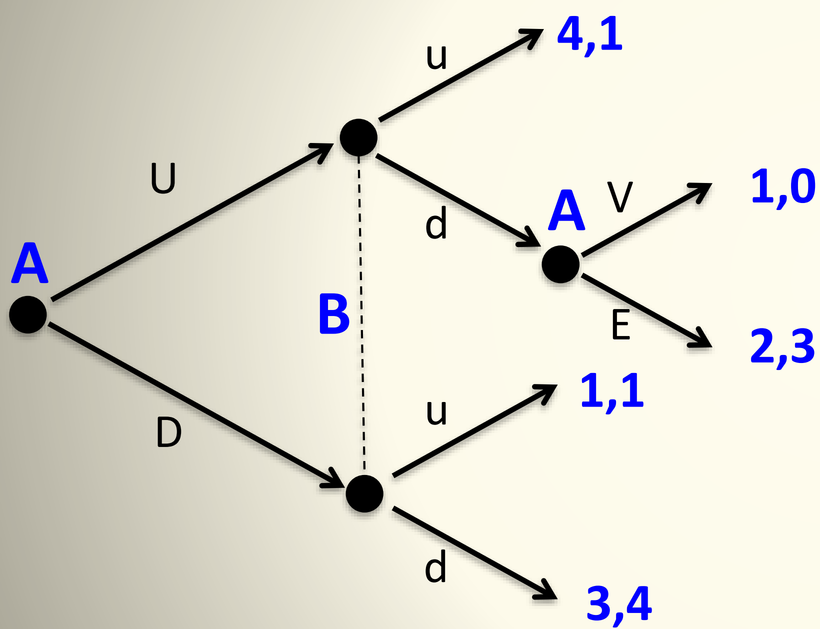
Role of information sets

- Many real political or social events may present phases of both **simultaneity** <u>and</u> **time dependent interaction**
- Through the concept of **information set** game theory can face **strategic interactions that are partly sequential and partly simultaneous** (which is the same as interactions made partly by visible and partly by hidden moves)

Strategies with imperfect information

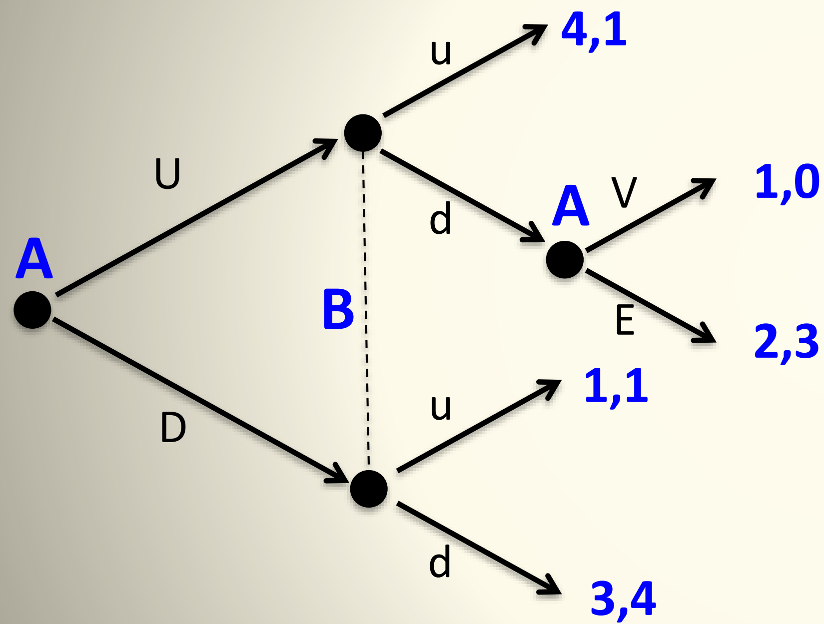
- To determine an equilibrium it is necessary to know the strategies of each player i (i = 1, 2, …, n)
- In a dynamic game with imperfect information…for any given player ***i* a strategy specifies what that player should do at any information set** (not at **any node** as in a dynamic game with perfect information, i.e., in a dynamic game with imperfect information, **one information set can involve more than one node**!!!)
- How to solve such game?

Example: strategies in the tree



- B has how many strategies?
- B is involved in just one information set (with two nodes) that involves 2 possible moves. Therefore it has 2 strategies:
- **{u , d}**
- A has how many strategies?
- It is involved in 2 information sets (each of them singleton), each with two possible moves. Therefore it has 4 strategies:
- **{UV , UE , DV , DE}**

Example: strategies in the tree



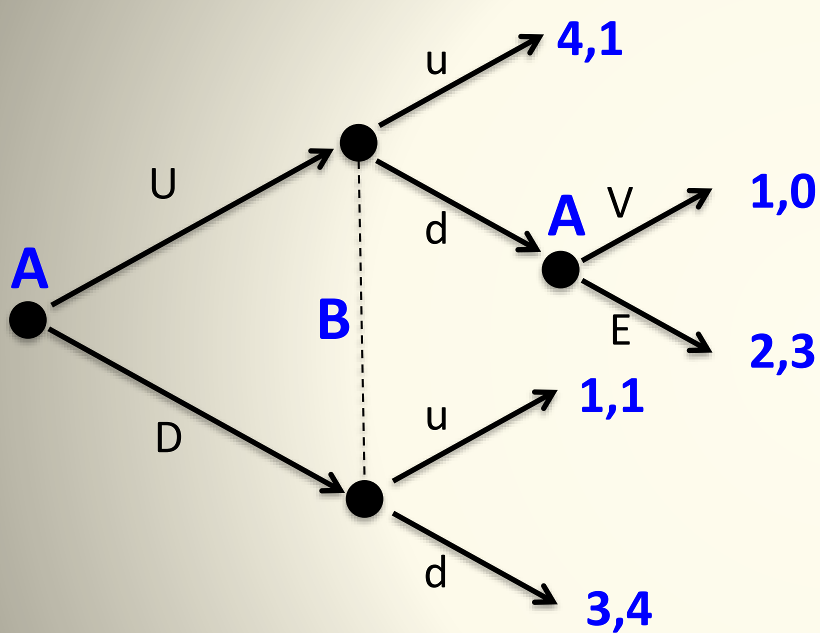
- B has two strategies:
- **{u , d}**
- A has four strategies:
- **{UV , UE , DV , DE}**
- Although **DV** and **DE** are sequences of moves never played by A, they cannot be discarded: remember the discussion about “credibility of threats (and promises)”. B decides her strategy based on what she expects player A would do after her choice, and this is anticipated by player A (backward induction!). Otherwise how would it be possible to identify any equilibrium?

How to solve such game?

- You have two options, that always PRODUCE to the same conclusion

1. Either you start to analyze the entire game and then you move to the (proper) subgame(s)

How to solve such game?
(1) From the tree to the matrix …

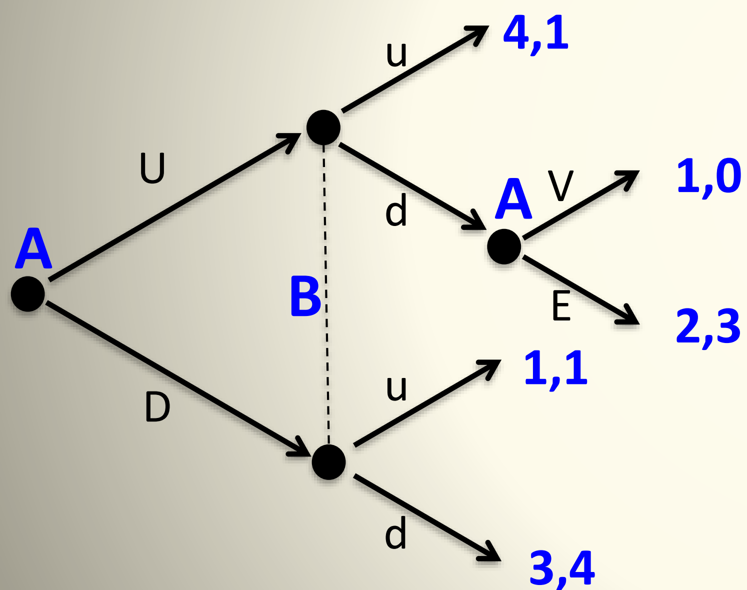


		B	
		u	d
A	UV	**4,1**	**1,0**
	UE	**4,1**	**2,3**
	DV	**1,1**	**3,4**
	DE	**1,1**	**3,4**

The matrix shows three NE: **{UV,u}, {DV,d}, {DE,d}**

How to solve such game? (1) … and back to the tree

- NE: **{UV,u}, {DV,d}, {DE,d}**
- Are all those equilibria **coherent with backward induction**?



- By backward induction if A has the opportunity to make her second move, she will choose E (2>1)
- {UV,d} and {DV, d} do not therefore satisfy subgame perfection!!!
- **➔ The only strategy profile coherent with backward induction is {DE,d} ➔ (3,4)**

Normal form and extensive form

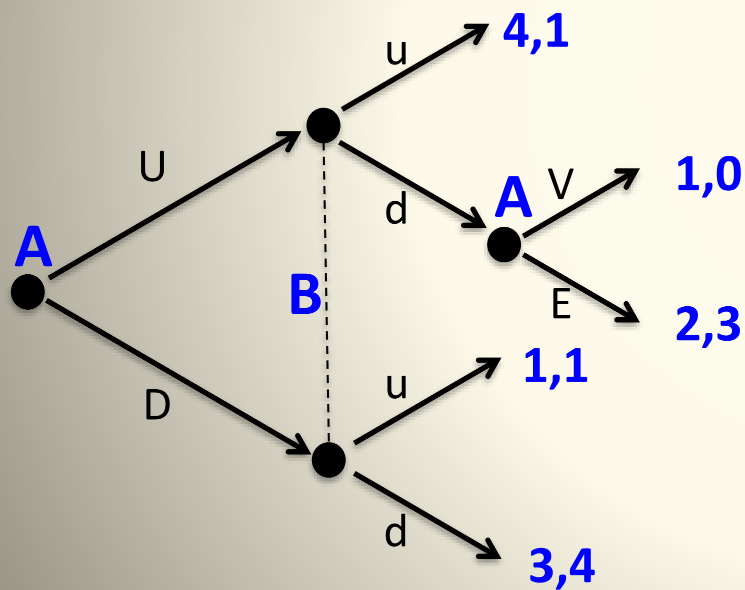
- At first sight controlling the strategies seems therefore the same as putting the game in **normal form**
- However not all equilibria in normal form **satisfy backward induction**
- Taking into account the sequence of moves, game trees (extensive form) give in fact information on games that matrices (normal form) **do not give** (remember the difference between NE and SPNE!!! And credible and NOT credible strategies!!!)
- **Trees and information sets give a more precise account than matrices of social interactions**

How to solve such game?

- You have two options, that always PRODUCE to the same conclusion

1. Either you start to analyze the entire game and then you move to the (proper) subgame(s)
2. Or you first solve the (proper) subgame(s) and then move to the remaining game

How to solve such game?
(2) Solve first the (proper) subgame(s)!



There are two subgames

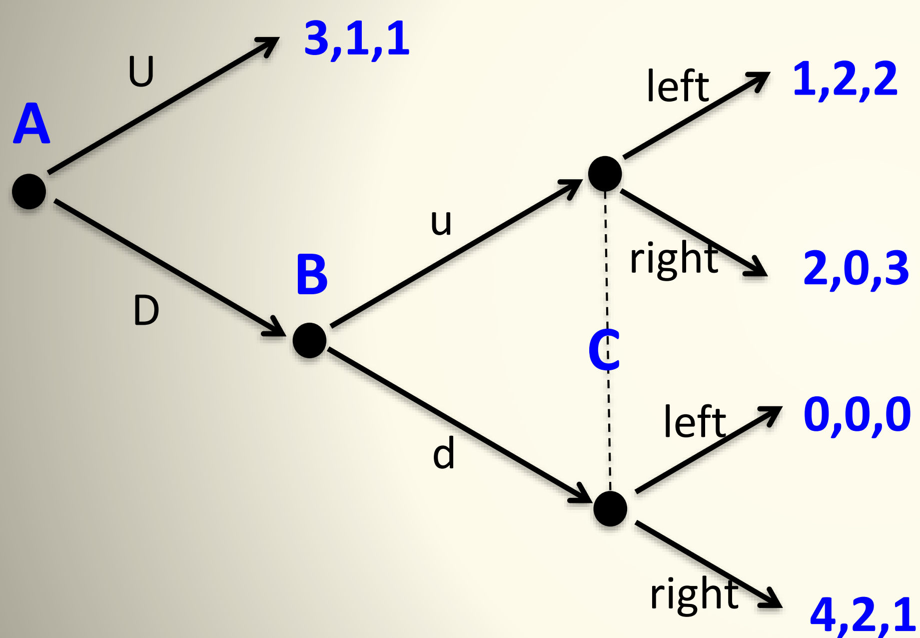
1. The first (the proper subgame) starts at the node where A moves for the second time
2. The second is the whole game

…. first solve the proper subgame…so that the entire matrix of the game simplifies as:

		B	
		u	**d**
A	**UE**	**4,1**	**2,3**
	DE	**1,1**	**3,4**

The only SPNE is {DE,d} ➔ (3,4)

A further example: three players



How can we represent a three players game in normal form?

Two matrices

A plays U			
		C	
		left	right
B	u	3,1,1	3,1,1
	d	3,1,1	3,1,1

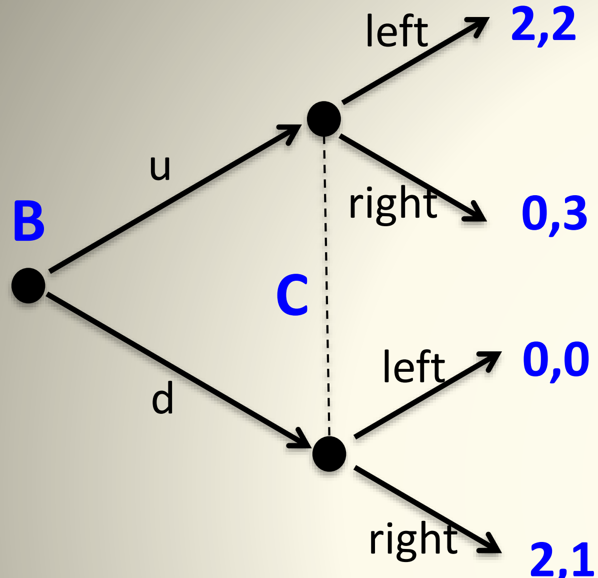
A plays D			
		C	
		left	right
B	u	1,2,2	2,0,3
	d	0,0,0	4,2,1

A: matrix player
B: row player
C: column player

The game has various NE, which ones? **{U,u,left}, {U,u,right}, {U,d,left}, {D,d,right}**

But is for example {U,u,left} a SPNE?

Consider the subgame where only players B and C play



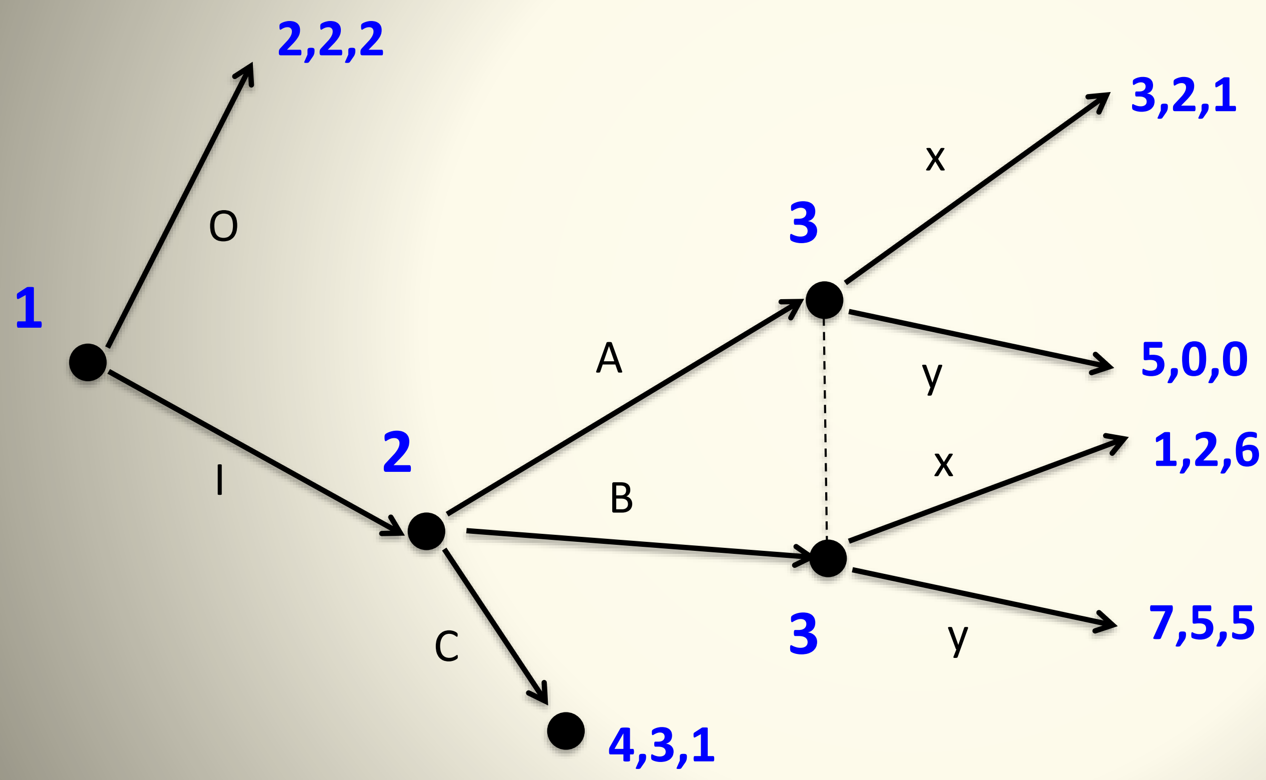
		C	
		left	**right**
B	**u**	**2,2**	**0,3**
	d	**0,0**	**2,1**

"right" is a dominant strategy for player C and the matrix of the subgame has the only NE **{d,right}**

That excludes **{U,u,left}** from being a SPNE

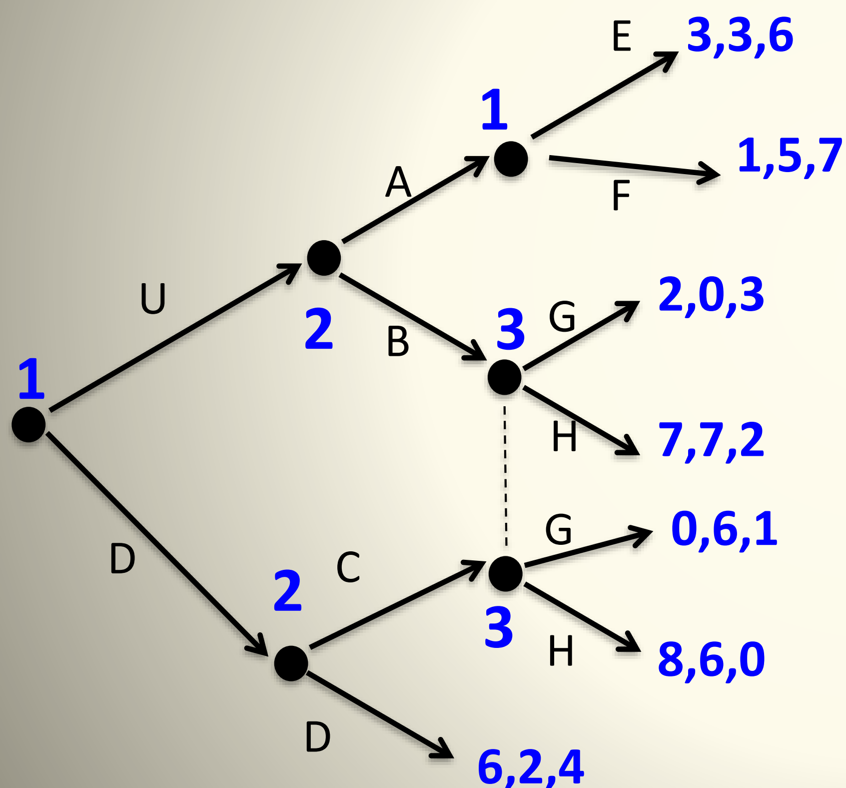
The only SPNE of the original game is **{D,d,right}** leading to the outcome **(4,2,1)**

A further example



How many players? Subgames? Strategies for each player?

A further example



How many players?
Subgames?
Strategies for each player?

To summarize the procedure to find out SPNE:

first solve the proper subgame(s) and then go back to the whole game or viceversa.

It should always produce the same result!

Home Exercise

The USA-IRAN example

- Three players: USA, Iran and opposition in Iran
- USA moves first and has to decide if killing or not an Iranian leader
- Then Iran has to react to that, observing the action played by USA: either respond to that “seriously”, or not respond to that “that seriously”
- Then USA observes it and must decide if increasing the pressure on Iran or otherwise
- The Opposition in Iran **does not observe** this decision of USA, but must decide if protesting or not

The USA-IRAN example

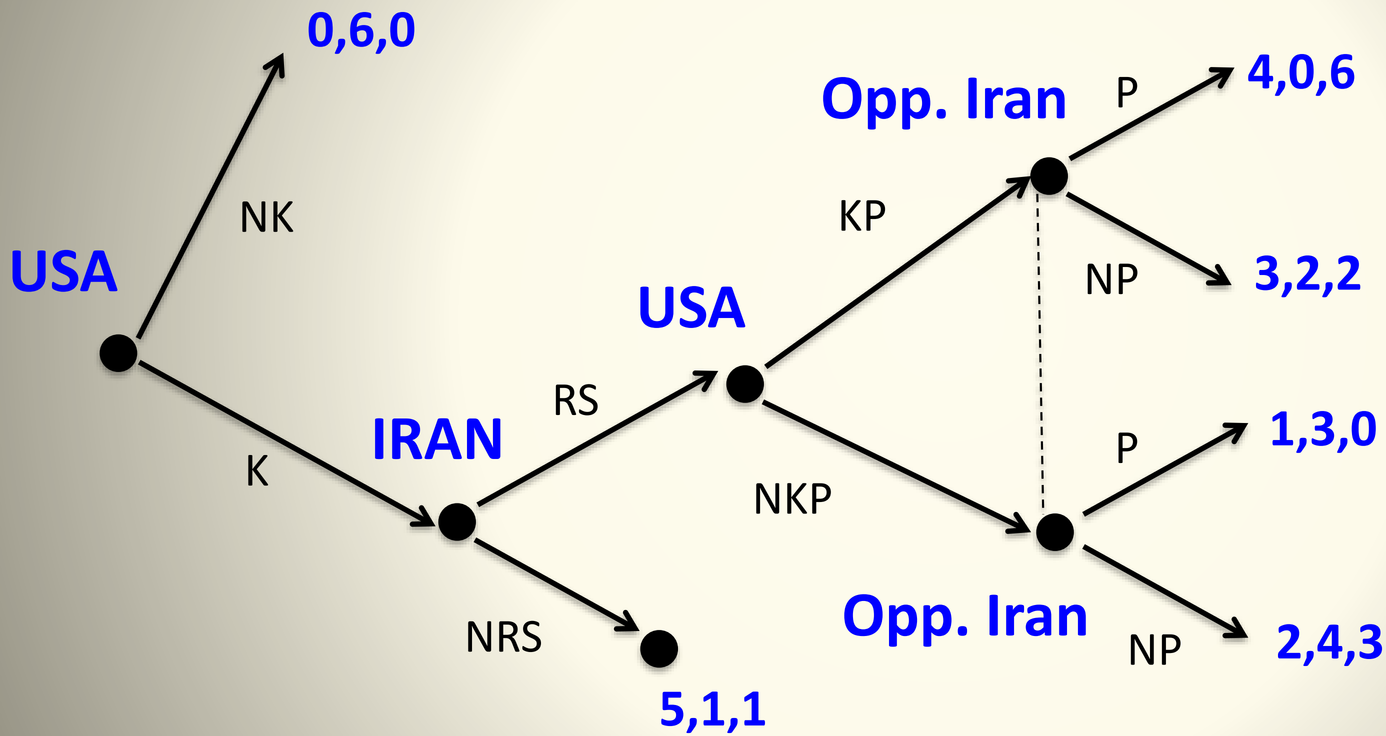
Our assumptions:

- The Opposition has to decide to protest or not only if Iran reacts seriously to USA killing of the Iranian leader
- The Opposition produces a **regime-change** only if it protests and USA keeps increasing its pressure
- A protest, without a USA backup, would mean the destruction of the opposition
- The Opposition would prefer not to protest when USA does not increase its pressure on Iran rather than not to protest when USA does increase its pressure (it would be a lost opportunity for the opposition not protesting in this latter case!)
- Iran would always prefer to react seriously to USA killing of Iranian leader, but when the Opposition is successful
- USA would prefer a not serious reaction to the killing of the Iranian leader to any alternative (and risky) scenario

The USA-IRAN example

any reference to real facts or persons is purely coincidental

The USA-IRAN example



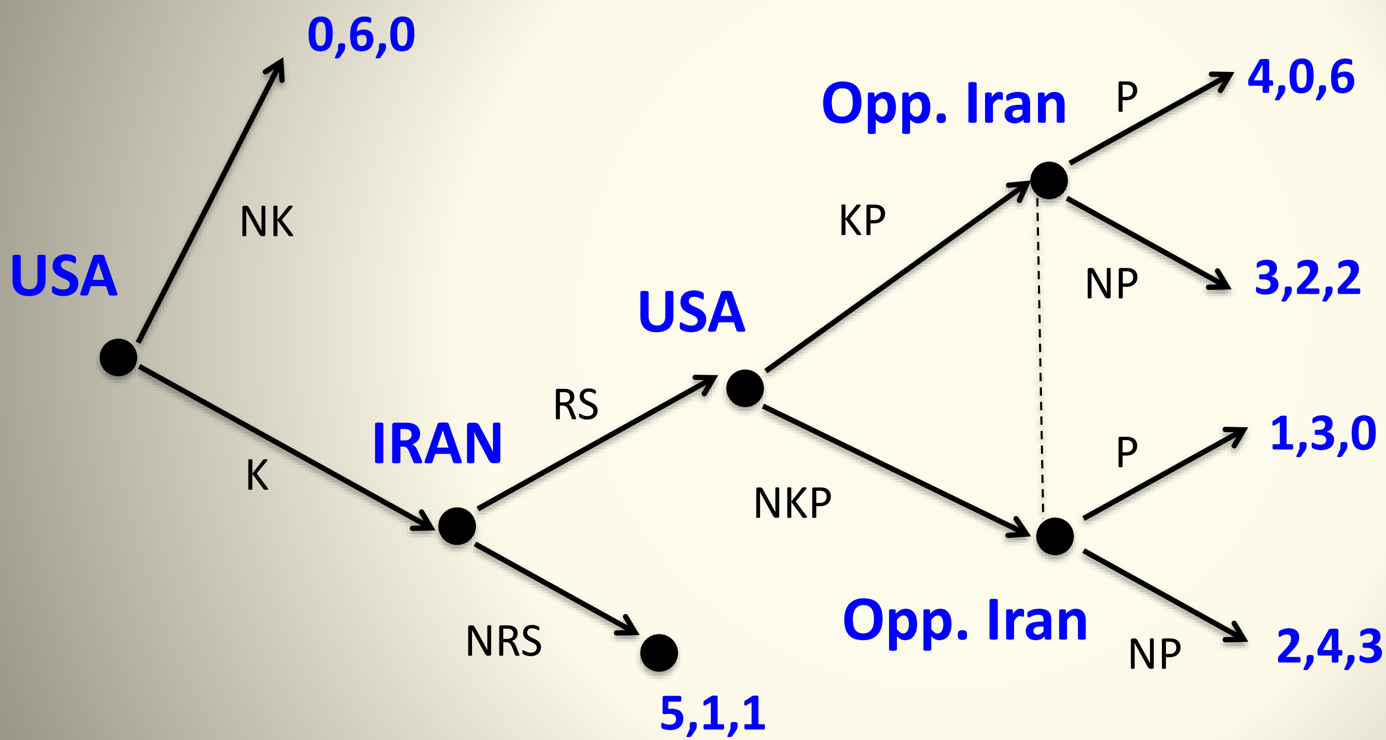
Labels for moves:

USA – NK (not killing); K (killing); KP (keep pressure); NKP (not keeping pressure)

IRAN – RS (reply strong); NRS (not reply strong)

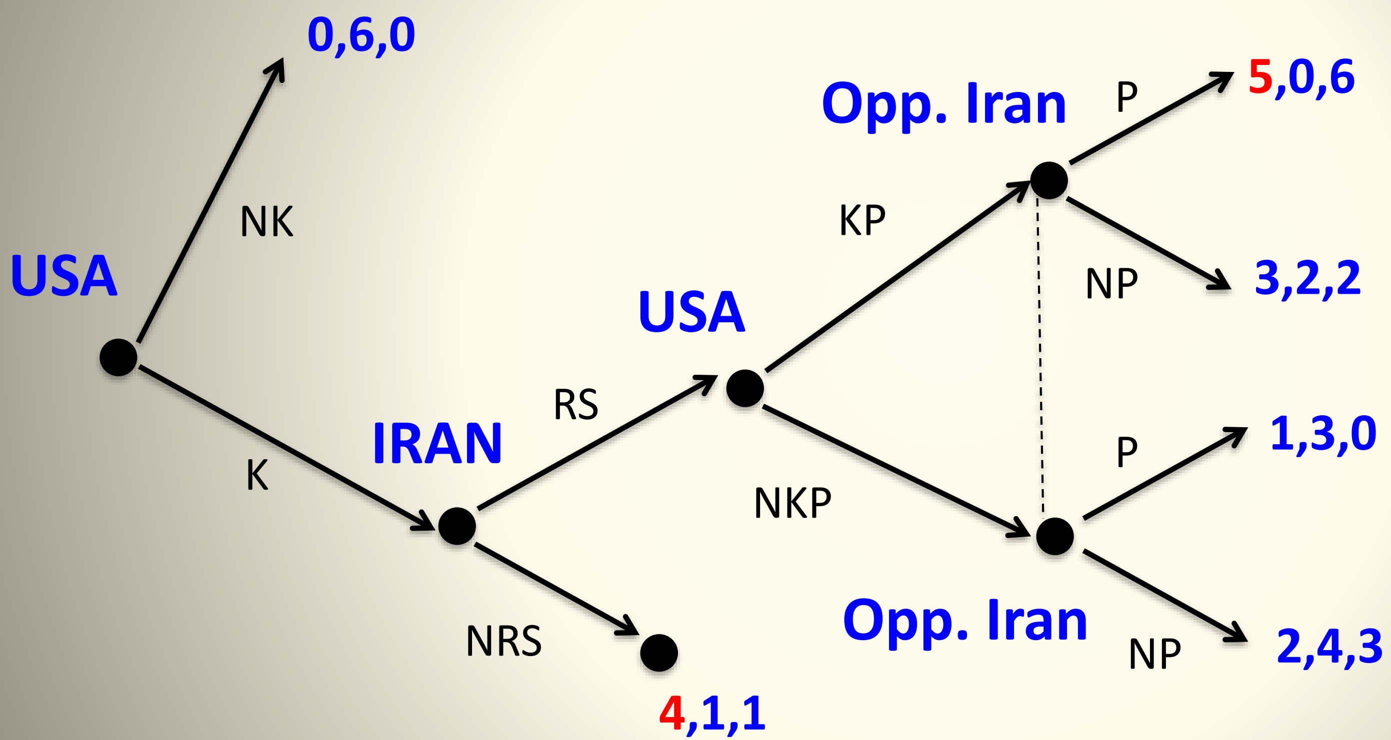
Opp. Iran – P (protest); NP (not protest)

The USA-IRAN example



How many players? Subgames? Strategies for each player?

The USA-IRAN example



Does it change anything if USA would prefer a regime-change to any other outcome?