Game Theory

7 – Games of Incomplete information: dynamic games

Incomplete information and estensive form

- Even in some dynamic setting, we can have different **types** but NO possibility to **update** beliefs by players according to actions undertaken during the game **whenever** the "type players" do not move FIRST!

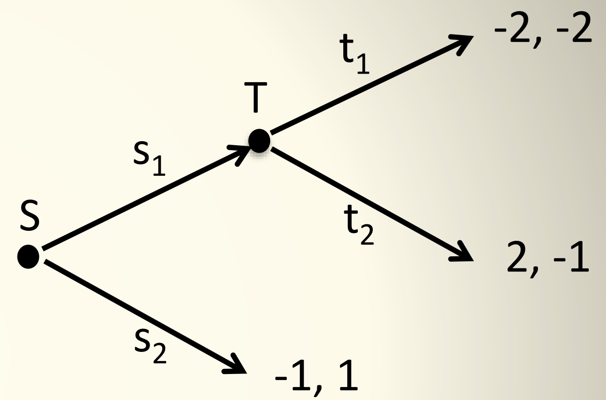
Incomplete information and estensive form

- Let us consider an electoral college where plurality rule is in effect
- In a multi-partisan system, party T is the usual winner presenting a popular local candidate
- Considering the opportunity to interrupt this tradition, party S may choose to present one of its national leader (s_1) or to give up (s_2)
- In case (s_1) is played, T may decide to answer presenting itself a national leader (t_1) or to continue with the local candidate (t_2)

"Choosing the candidate" game (1)

For **T** the best outcome is the **status quo** (it wins with no costs)
Behaving **as usual** against the challenge is the second (the contest is uncertain but a national leader is saved for other situations)
The last result is **accepting the challenge** (the result is uncertain but a national leader is lost for other situations)

For **S** the best result is **challenging** with no reaction by T (high probability of winning)
The second best is **not to challenge** at all (it loses but the national leader is saved
The last result is when its **challenge is accepted** (no better chance to win and one national leader lost)



The solution it immediate by backward induction:

- SPNE (s_1 ; t_2)
- The game develops with a challenge ignored
- The outcome is (2,-1)

"Choosing the candidate" game (2)

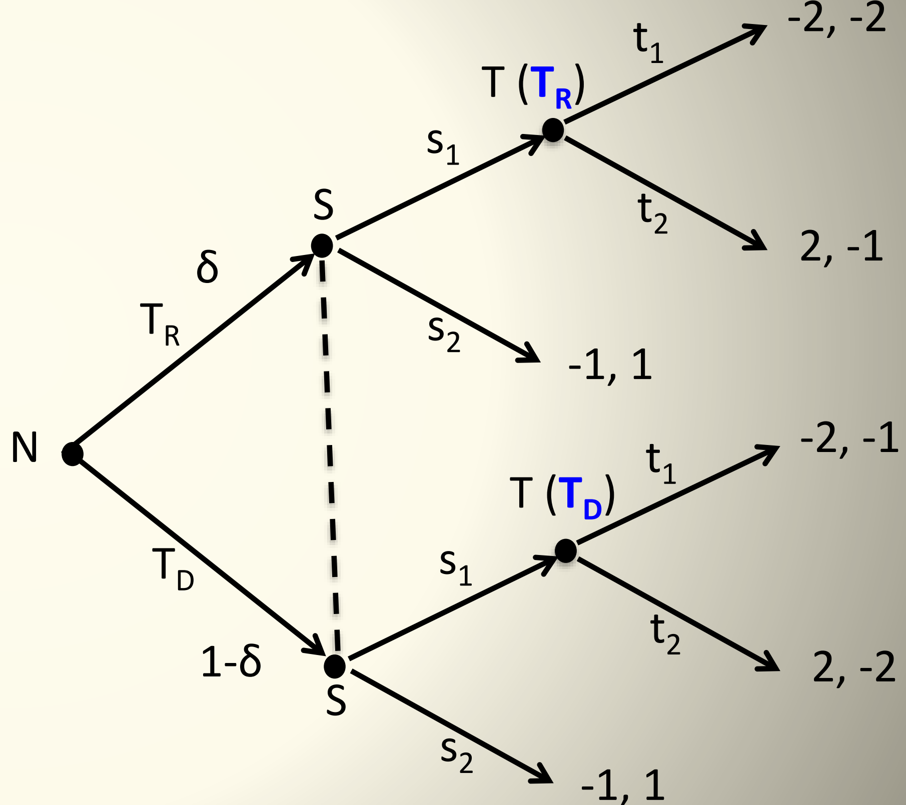
Let us **now** suppose that the challenging party S is uncertain about party T preferences

1. T is considered **preferring to lose** the college than to spend a national leader
2. Or T is considered **preferring to win** the college than to go without a leader in other competitions

In other words T may be represented by **the type T_R** (ready to loose the college) or by **the type T_D** (determined to get the seat)

Uncertainty can be represented by an **initial move of the Nature** choosing the type T_R (with probability delta δ) or T_D (with probability 1- δ) of party T

N's move is private information of player T

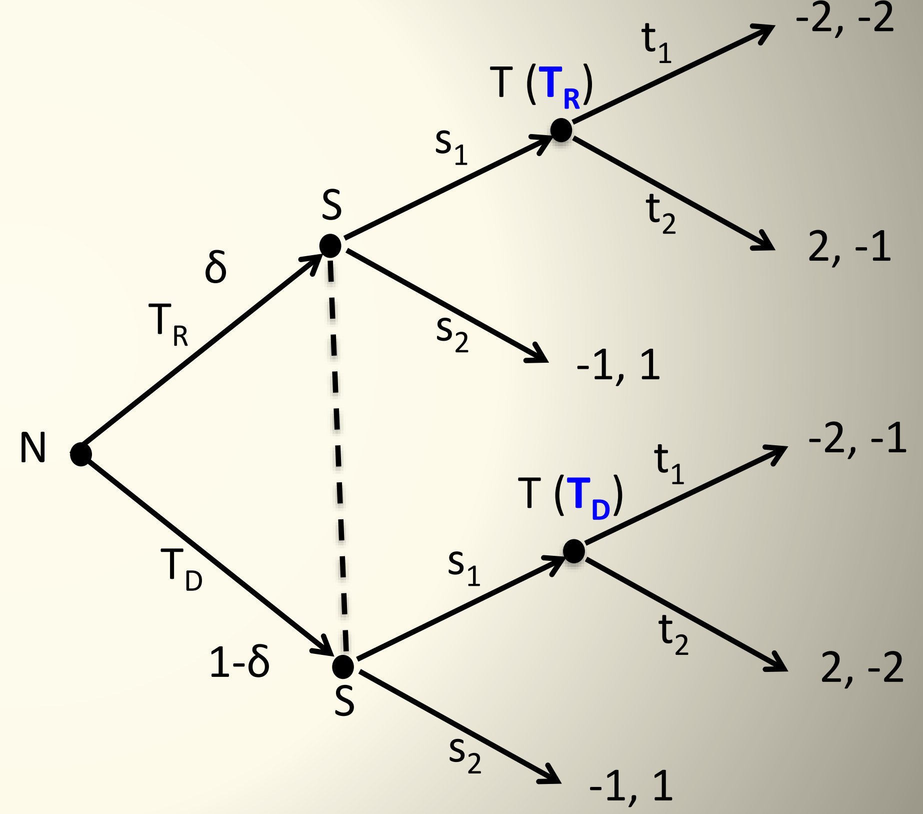


"Choosing the candidate" game (2)

The game is among three types of players: S, T_R and T_D

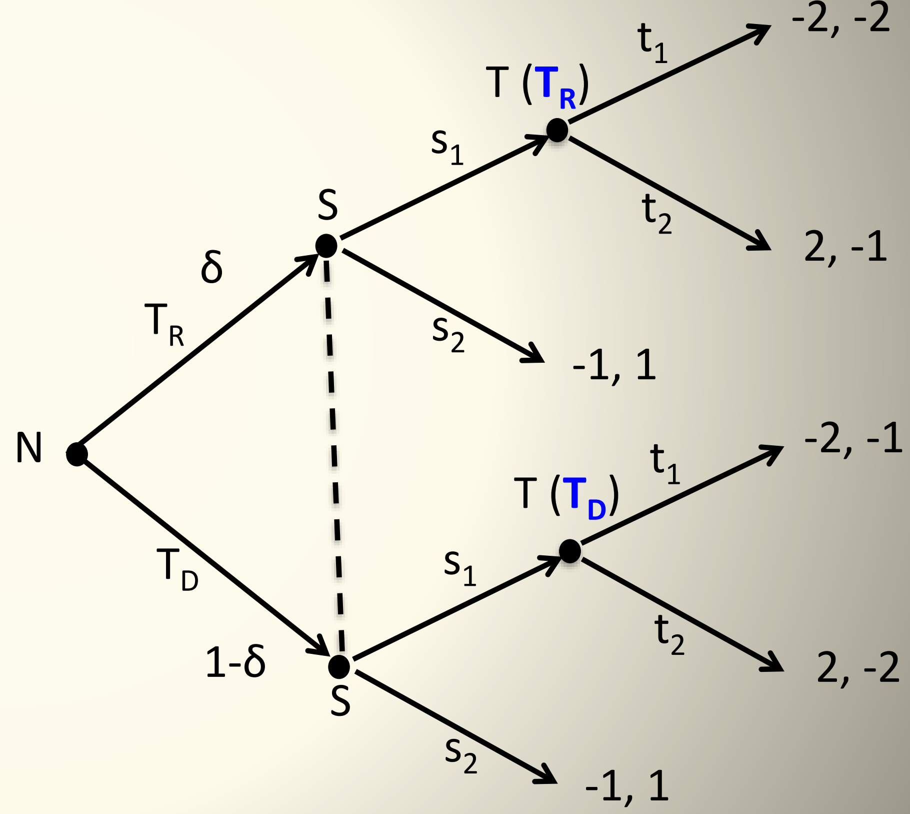
How many subgames do you have? How many strategies are available for each type of player?

Each of the two t type is involved in one proper subgames, therefore we can apply **backward induction**!



“Choosing the candidate” game (2)

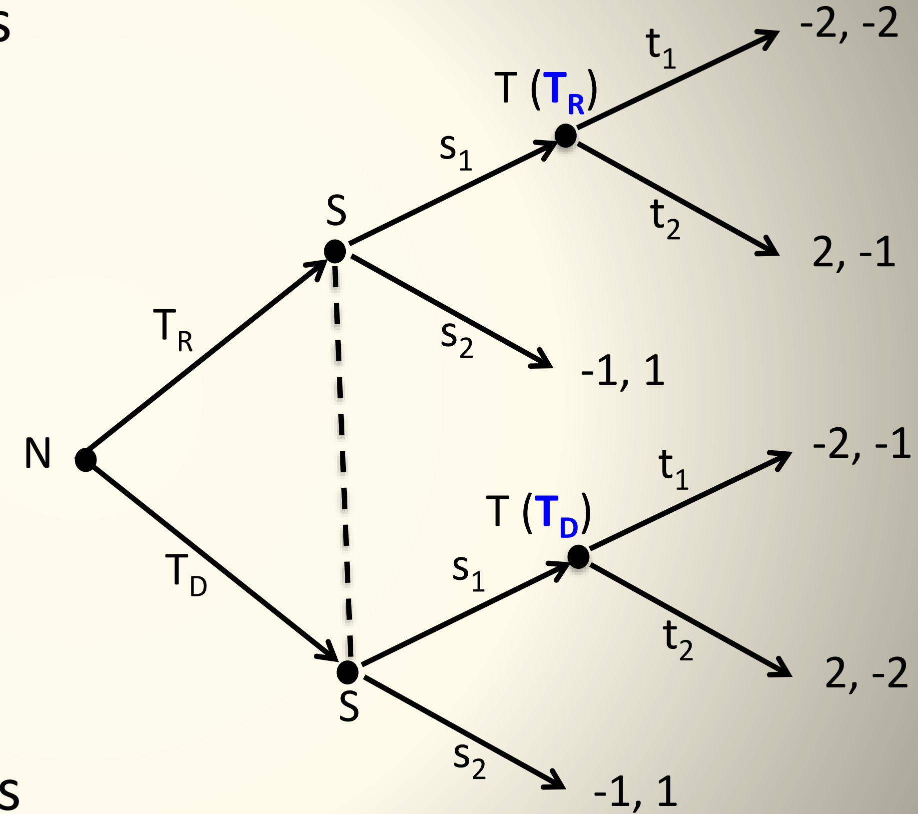
Let’s solve the game by applying first the already discussed conditions ***[2]*** and ***[3]*** in the case of a static Bayes game



"Choosing the candidate" game (3)

There are eight strategy profiles that are:
(s_1,t_1,t_1); (s_1,t_1,t_2); (s_1,t_2,t_1); (s_1,t_2,t_2); (s_2,t_1,t_1); (s_2,t_1,t_2); (s_2,t_2,t_1); (s_2,t_2,t_2)

However, playing t_1 for T_R or t_2 for T_D wouldn't be consistent with backward induction (conditions ***[2]*** and ***[3]!!!***)
Therefore…only the two profiles **(s_1,t_2,t_1)** and **(s_2,t_2,t_1)** survive



"choosing the candidate" game (4)

Let's go back to player S and let's apply condition ***[1]***. In this case the probabilities δ that S assigns to T_R and $1-\delta$ that S assigns to T_D are unknown

We need to see if values exist of δ such that one or the other survived profiles **(s_1,t_2,t_1)** and **(s_2,t_2,t_1)** are not dominated for S

The expected utilities of party S for the two profiles are

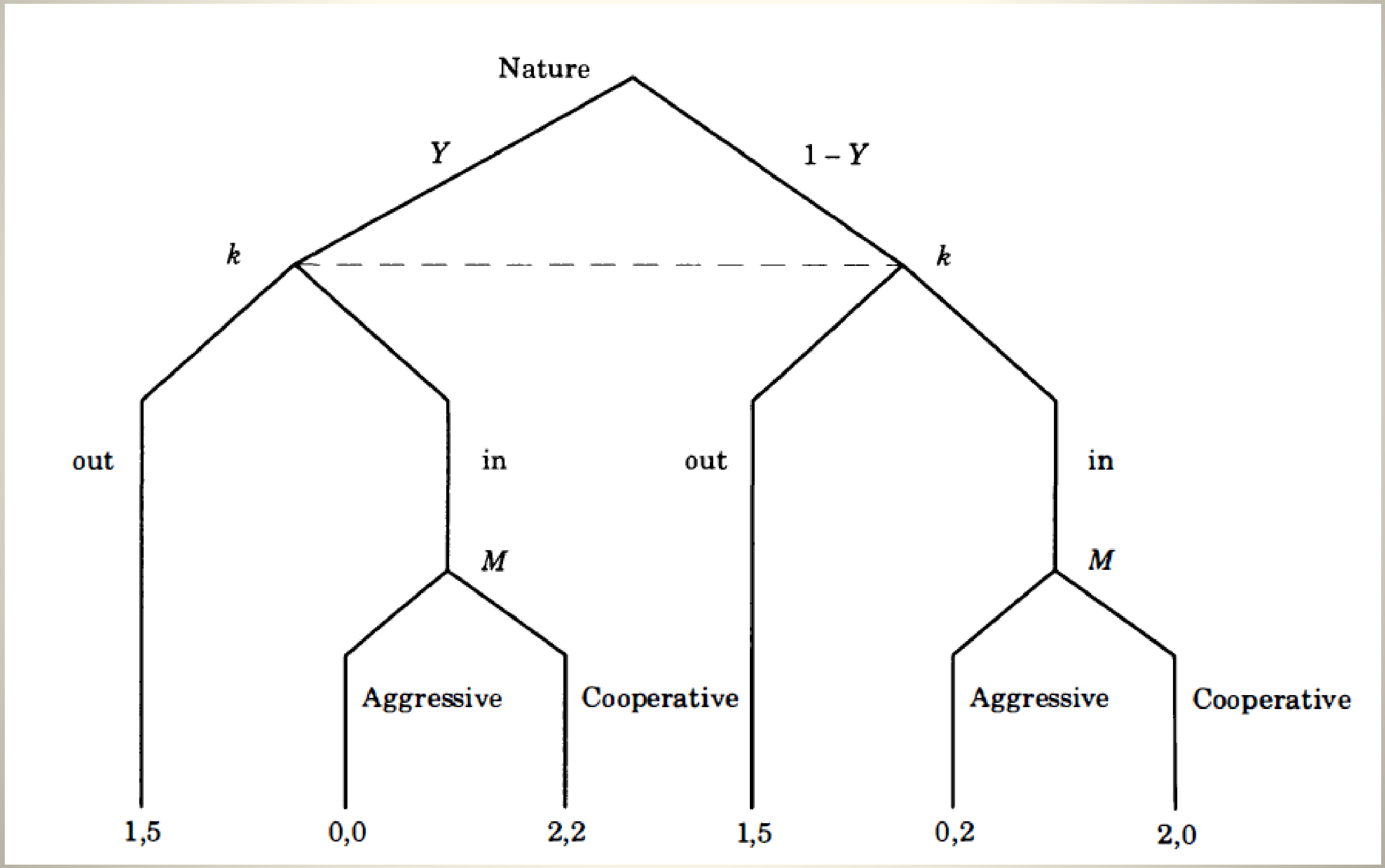
$$Eu_s(s_1,t_2,t_1) = 2\delta-2(1-\delta) = 4\delta-2$$
$$Eu_s(s_2,t_2,t_1) = -\delta-(1-\delta) = -1$$

➔ $Eu_s(s_1,t_2,t_1) > Eu_s(s_2,t_2,t_1)$ if and only if $\delta > ¼$

➔ The game has two BNE equilibria:

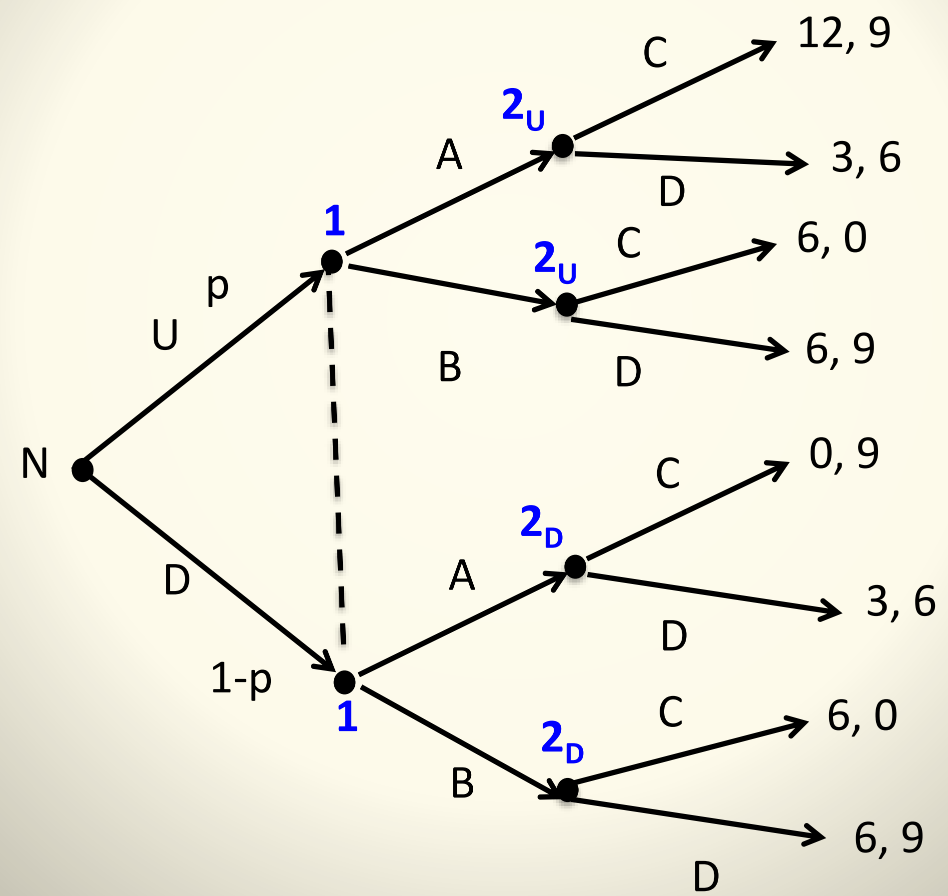
1. **$\{(s_1,t_2,t_1)$, $\delta > ¼$**$\}$ (when S believes that the incumbent party has at least 25% probability of being T_R , i.e. disposed to lose the college, S runs a national leader and T answers accordingly to its character)
2. **$\{(s_2,t_2,t_1)$, $\delta < ¼\}$** (S continues to present the local candidate and T reacts as before, if S believes that the incumbent party has at least 75% probability of being T_D , i.e. she is resolute to keep the college seat)

Same example with different payoffs (i.e., Nature defines M types)



Two BNEs: (in, c, a) with y>1/2 and
(out, c, a) with y<1/2

Home exercise (Nature defines player 2 types)



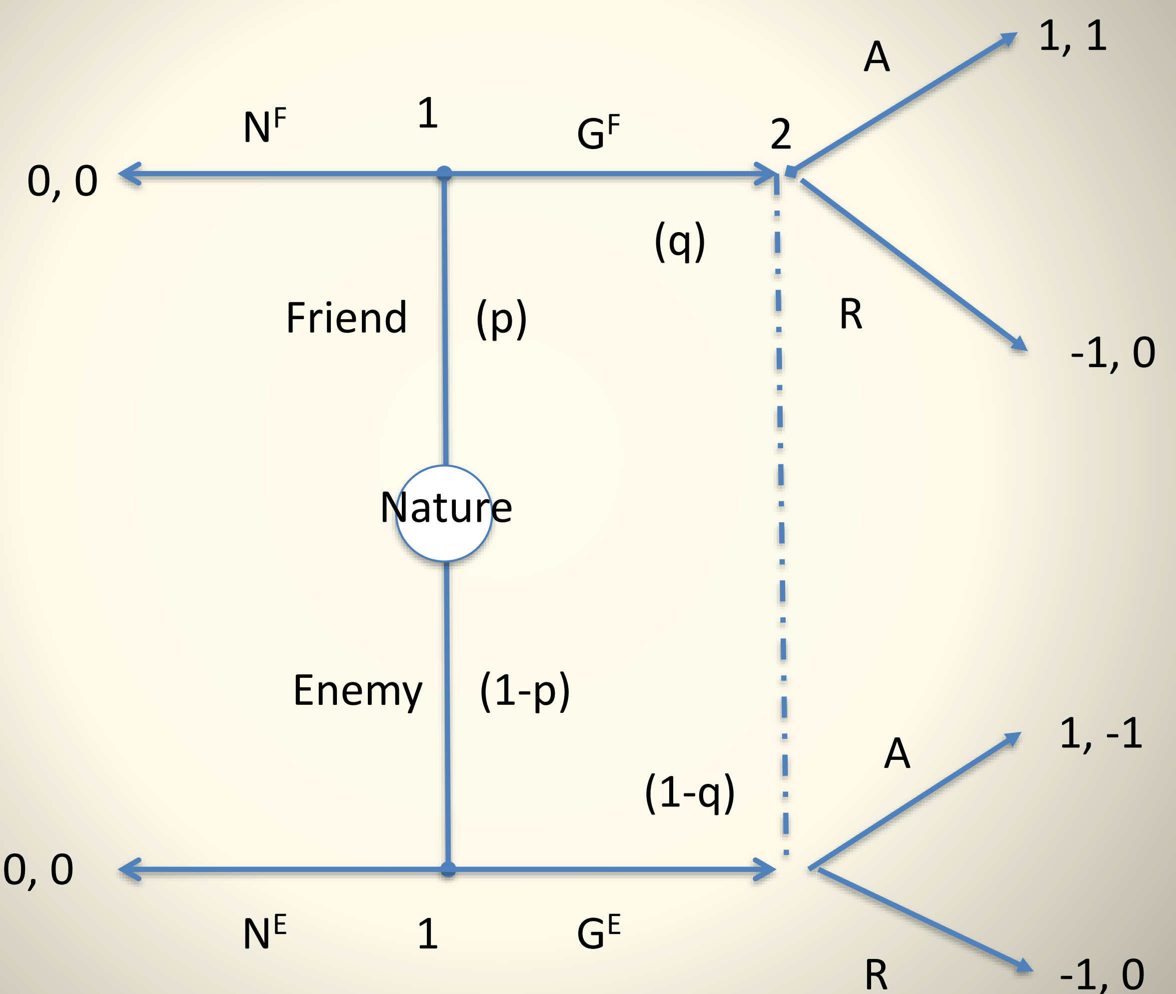
Bayesian games with updating of beliefs

Cinema time!

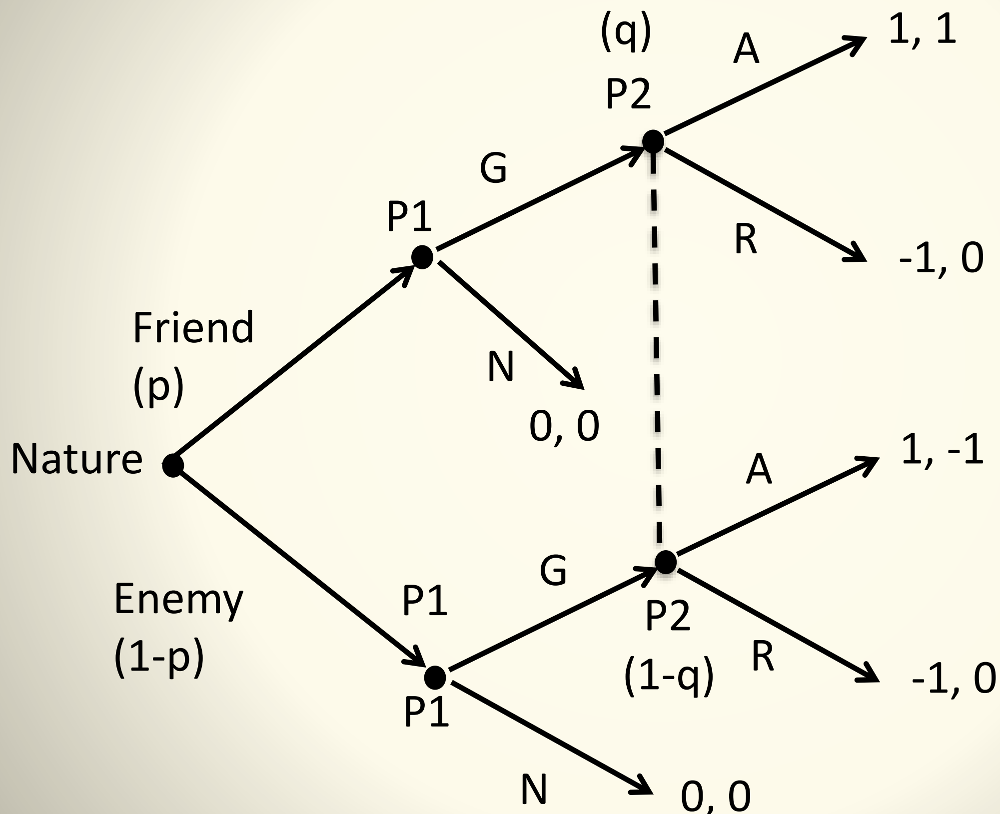
Bayesian games with updating of beliefs

- Consider the following game: **the gift game**. *Friend* tends to keep desirable objects in his pocket to offer you as a gift, the *Enemy* no (such as rocks or frogs…). In this variant of the game, player 2 prefers to accept a gift **only** from a Friend
- Player 1 however can be of two different types: Friend or Enemy

Bayesian games with updating of beliefs



Equivalent graphical representation



Bayesian games with updating of beliefs

- Each type of player 1 has 2 strategies available (offering or not offering a gift)…
- …while player 2 has 2 strategies available at his/her only information set (i.e., accepting or not accepting the gift)
- This game is a dynamic one where player 2 **can update his/her beliefs** about player 1 type according to what player 1 does…

Bayesian games where the updating of beliefs is possible

- There is **indeed** a difference, in this game, between **p** (i.e., the initial belief about player 1's type) and **q** (player 2's updated belief about player 1's type, **after that player 2 observes** the strategy of player 1)
- Here **q** is the probability that a Friend **is giving you** a gift
- For example, suppose that player 1 behaves according to strategy (N^F, G^E)
- As a result, according to such strategy by player 1, player 2 now expects a gift from an enemy with (an updated) probability equal to 1, i.e., p=q=0

Bayesian games where the updating of beliefs is possible

- In general, player 2 update always his belief about player 1’s type, **conditional on arriving at player 2’s information set - given player 1’s strategies** (that is, conditional on receiving a gift in our example)
- How to include such a possibility into an equilibrium?

A Perfect Bayesian Equilibrium

- PBE is a solution concept that **incorporates** *sequential rationality* and *consistency of beliefs*
- *Sequential rationality* requires that players maximize their payoffs from each of their information sets (**on** or **off** the equilibrium path! More on this later…)
- How to reach that? *Consistency of beliefs*! In a PBE player's updated beliefs should be **consistent** with Nature's probability distribution **and** other player's strategy
- In general consistency between Nature's probability distribution (**p** in the previous example), player 1's strategy, and player 2's updated belief (**q** in the previous example) can be evaluated by using **Bayes rule**

Bayes rule

- Bayes rule gives the **conditional probability** of an event when another event has been observed, i.e., it gives us a criterion to determine **how new information should change our beliefs about a given event**

Bayes rule

- Let p(A) and p(B) two a-priori probability of the different events A and B
- Let us write **p(A|B)** the probability of the event A when B has been observed
- **Bayes rule is a formula for determining p(A|B)**
- More formally:
- $p(A|B) = (p(A)\, p(B|A))/[(p(A)\, p(B|A) + p(\neg A)\, p(B|\neg A)]$

where: p(A) is the a priori probability of A before occurring B, p(B|A) is the conditional probability of B given A, p(¬A) is the a priori probability of ¬A and p(B|¬A) is the conditional probability of B given the event “not-A”

Bayes rule

- **An example:**
- You are on the train and you want to understand if the person sitting next to you is a centre-right voter
- You know a priori that 54% of the Italian citizens are centre-right voters (46% centre-left)
- Now the person sitting next to you open a newspaper. You know that the 35% of centre-right voters read that newspaper (while it is read by 65% of centre-left voters)
- Which is your update belief that the person sitting next to you is a centre-right voter?
- $p(CR|N) = p(CR)\ p(N|CR)/[(p(CR)\ p(N|CR)+p(CL)\ p(N|CL))] =$

$=.54*.35/(.54*.35+.46*.65) = .387$

Bayes rule

Bayes rule:

- More in general, given $(h_1, h_2,\dots, h_n)$ a set of mutually exclusive and exhaustive events compatible with the event k, then:

$$p(h_1|k) = \frac{p(h_1)p(k|h_1)}{\sum_{i=1}^{n} p(h_i)p(k|h_i)}$$

Bayes rule

- Going back to previous game:
- Suppose that the probability to meet a Friend determined by Nature is ½
- Let's further suppose that the two types of Player 1 adopt the following strategy (N^E, G^F)
- Before that strategy, p(FRIEND)=1/2. Now the update probability of p(FRIEND|G) is…
- p(FRIEND|G) = (p(FRIEND) p(G|FRIEND))/[(p(FRIEND) p(G|FRIEND) + p(ENEMY) p(G|ENEMY)]…that is:
- p(FRIEND|G) = (0.5 * 1)/(0.5 * 1 + 0.5 * 0) = 1

A Perfect Bayesian Equilibrium: definition

- Consider a **strategy profile** for the players (i.e., types), as well as **beliefs over the nodes at all information sets**. These are called a **PBE** iff:

1) each player's strategy specifies optimal actions, **given the strategies of the other players and her beliefs** (this is the same requirement of a BNE!)

2) the beliefs **are consistent** with Bayes rule *wherever possible* (?!? Be patient and you will understand it…)

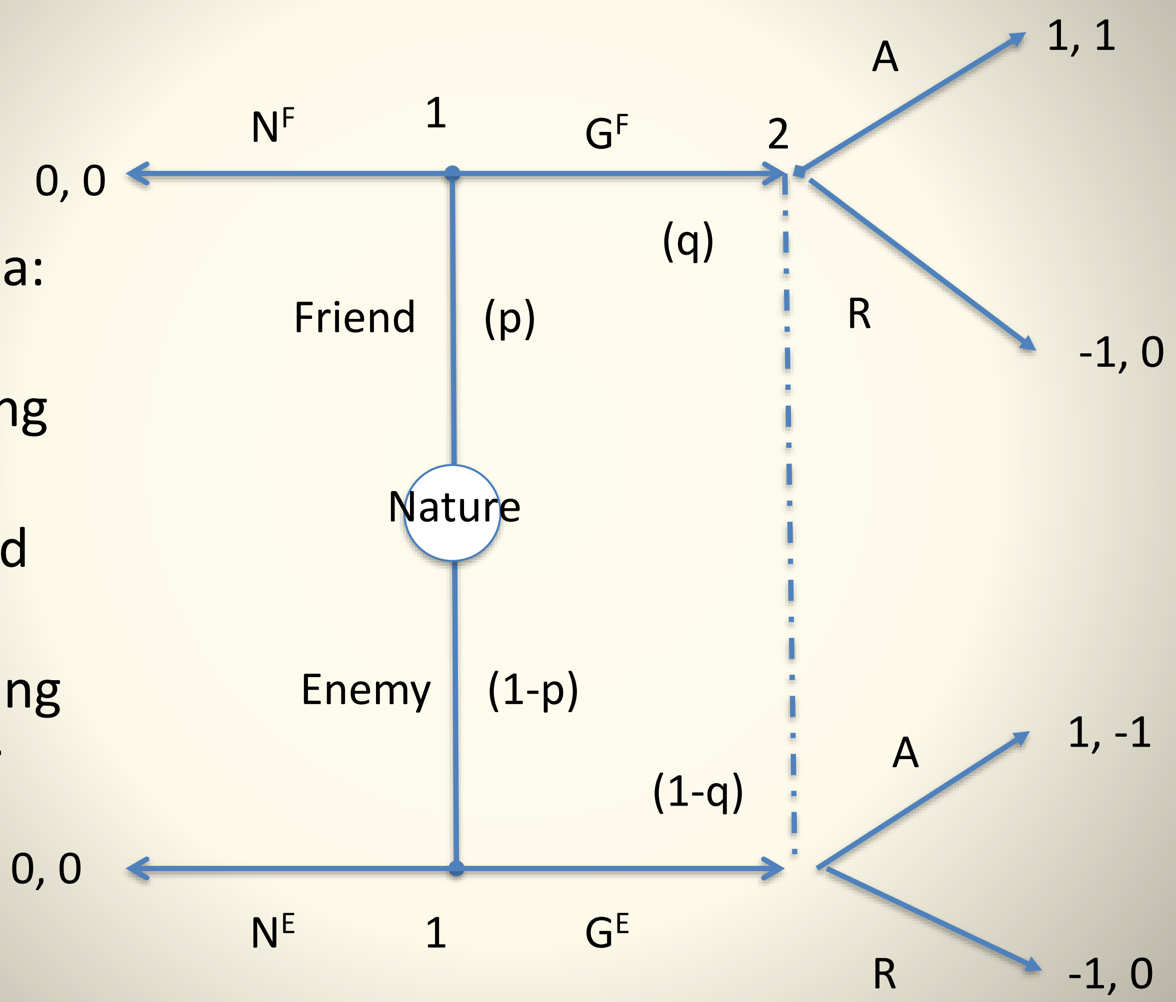
- In essence a PBE is a **coherent story** that describes **beliefs and strategies** in a game

A Perfect Bayesian Equilibrium: how to find it!

- Two additional terms are useful: we call an equilibrium as a **separating** one if all the types of a player behave differently
- We call an equilibrium as a **pooling** one if all the types behave the same

An application

There are **four** potential equilibria: two <u>separating</u> equilibria (featuring strategy $G^F N^E$, or strategy $N^F G^E$) and two <u>pooling</u> equilibria (featuring strategy $N^F N^E$, or strategy $G^F G^E$)



A Perfect Bayesian Equilibrium: how to find it!

– **Steps for calculating PBE:**

- Starts with a strategy for player 1 (in this case 2 strategies for the 2 types of player 1)
- If possible, calculate updated beliefs (**q** in the example) for player 2 by using Bayes rule. In the event that Bayes rule **cannot be used**, you must **arbitrarily select an updated belief**; here you will generally have to check **different potential values** for the updated belief with the next steps of the procedure;
- Given the updated beliefs, calculate player 2’s optimal action
- Check whether player 1’s strategy is a best response to player 2’s strategy. If so, CONGRATULATIONS: you have just found a PBE!

An application

Let’s apply our procedure:

Separating with N^F G^E:

- given this strategy for player 1, it be must be that q|G=0 (Bayes rule!). Thus, player 2’s optimal strategy is R. But then the enemy type of player 1 would strictly prefer not to play G^E given that is not the best reply to R! The best reply to R would be for the enemy type N^E !
- Therefore, there is **no PBE** in which N^F G^E is played

An application

Separating with G^F N^E:

- given this strategy for player 1, it be must be that q|G=1 (Bayes rule!). Thus, player 2's optimal strategy is A. But then the enemy type of player 1 would strictly prefer to play G^E rather than N^E

✓ Therefore, there is **no PBE** in which G^F N^E is played

An application

Pooling with G^F G^E:

- here Bayes rule requires that q|G=p, so player 2 optimally selects A iff q=p>1/2. When q=p>1/2 there is therefore a PBE in which q=p and (G^F G^E, A) is played - PBE: (G^F G^E, A) , q=p; p>1/2
- **Why an equilibrium?** Given the strategy (G^F G^E) played by player 1, the best reply for player 2 to that GIVEN the belief specified (q=p; p>1/2) is A. And given the strategy adopted by player 2 (A), the strategy (G^F G^E) is the best reply to that for both players!

An application

Pooling with G^F G^E:

- On the other hand, in the event that q=p<1/2, player 2 must select R, in which case neither type of player 1 wishes to play G in the first place

✓ Thus there is **no PBE** of this type when q=p<1/2

An application

Pooling with G^F G^E:

- **But what will happen if q=p=1/2?**
- Then player 2 will be indifferent between playing A or R, so he will be mixing
- That implies looking for a mixed strategy PBE. That's something we won't discuss in this course

An application

Pooling with N^F N^E:

- In this case Bayes rule **does not determine** q
- Why? Cause in this case both types of player 1 play N, and player 2 **cannot update** q according to Bayes rule, given that G is not played and his information set is not reached **on the equilibrium path**!!!

An application

Pooling with N^F N^E:

- Still, regardless of player 1's strategy, player 2 will have some updated belief q at his information set
- This number has **meaning** even if player 2 believes that player 1 adopts the strategy N^F, N^E
- In this case, q represents player 2's belief about the type of player 1 when the "surprise" of a gift occurs (i.e., **off the equilibrium path**)

An application

- **Pooling with N^F N^E**:
- What would do player 2 in this eventuality? She will estimate her expected utility by playing either A or R given p
- This would lead to opt for R, iff $q<1/2$
- In other words, in order for R to be chosen, player 2 must have a sufficiently **pessimistic belief** regarding the type of player 1 after the "surprise" in which a gift is given
- Strategy R is therefore optimal as long as $q<1/2$. But given strategy R, N^F N^E is a best response to that by the two types of player 1!
- Note that without specifying what player 2 would do in the (surprising) eventuality she gets a gift, you cannot check if N^F N^E is actually an optimal strategy for both types of player 1!

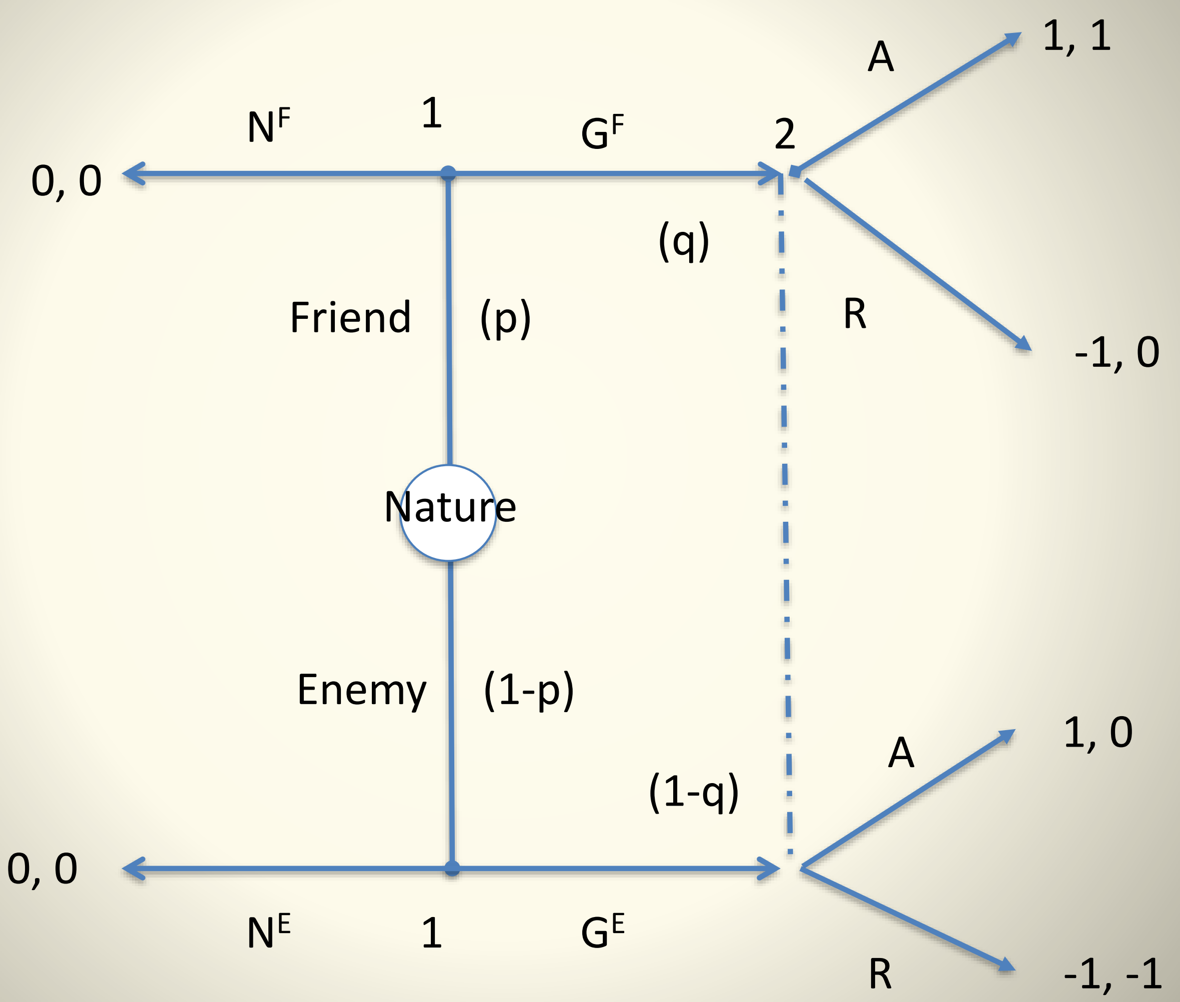
An application

- **Pooling with N^F N^E**:
- Thus, for q<1/2 there is a PBE in which player 2's belief is p and the strategy profile (N^F N^E, R) is played
- In this equilibrium player 2 believes that an **eventual (off-the-equilibrium path) gift signals** the presence of the enemy (a misanthrope?)
- PBE: (N^F N^E, R) , q<1/2
- On the other hand, if q>1/2 player 2 would select A. But then bother types of player 1 would have an incentive to switch their strategy! No PBE!

Another example

- Consider once again **the gift game**. However, in this variant of the game player 2 always prefer opening gifts than not opening it (*satisfying curiosity is a lovely gift by itself!*) that is…
- …R is a dominated strategy for player 2…

The Gift Game part 2

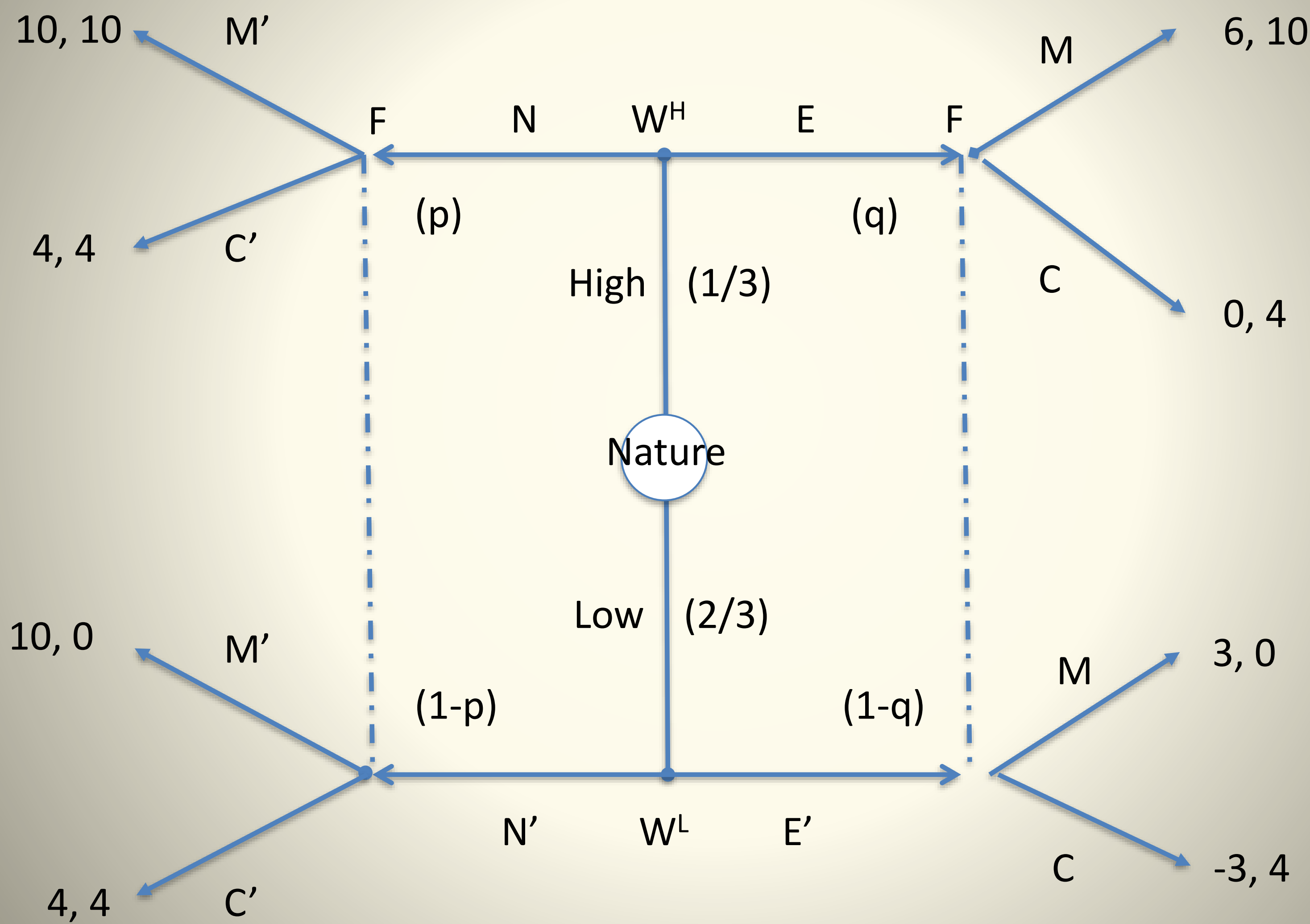


Only one BNE: (G^F G^E, A) , q=p; for any value of p

Job-Marketing Signaling

- **The signaling role of education**: which role of **formal education** in the **marketplace**?
- A worker (W) and a firm (F). The worker can be of **two types**: **high** or **low type**. Firm must decides whether to employ the worker in an important managerial job (M) or in a much less important clerical job (C). M produces a benefit of 10 to both types of worker, however they have **different education costs** (in terms not only of monetary costs, i.e., grants, etc.; but also in terms of opportunity costs): the high type to get an education must pay 4 units of utility, the low type 7. C produces a benefit of 4
- Importantly, education is of **no direct value to the firm**; the firm's payoff does not depend on whether the worker gets an education, but **only** on the intrinsic type of the worker

Job-Marketing Signaling



Job-Marketing Signaling

- Each type of worker has 2 strategies available (getting or not an education), while the firm has 2 strategies available at each of its 2 information sets (i.e., always offering a managerial job; always offering a clerical job; etc.)
- The initial system of beliefs of the Firm given by Nature is:
  High Type=1/3; Low Type=2/3

Job-Marketing Signaling: comments

- Two PBNE: the first one is (EN’, MC’, p=0, q=1)
- **Insights:**
  - **First**: the only way for the high-type worker to get the job that she deserves is **to signal her type** by getting an education. Otherwise the firm judges the worker to be a low type
  - **Second**: the value of education as a signaling device depends on the types’ **differential education costs**, not on any **skill enhancement** that education deliver
- That is…to the extent that highly productive people are **more likely** than less-productive people to get degrees, than rather than helping people become smart, universities exist merely to help people who are already smart **to prove** that they are smart!

Job-Marketing Signaling: comments

- This also means that **highly demanding** Universities and Professors are making you a favour given that reinforce the signal you are sending to the market! (this is something impossible to make understand to several ppl…)
- Similarly, **free universities** risk to depress the credibility of the signals sent by getting an education (given that they greatly reduce the education costs, so everyone could get a degree; but then, why getting it if it is not useful ex-post as a signalling device? Indeed no PBNE with EE'!!!)

Job-Marketing Signaling: comments

- Two PBNE: the second one is (NN', CC', p=1/3, q<2/5)
- Why p=1/3? Bayes Rule!
- p(high|N) = (p(high) p(N|high))/[(p(high) p(N|high) + p(low) p(N|low)]…that is: p(high|N) = (1/3 * 1)/(1/3 * 1 + 2.3 * 1) = 1/3
- **Insights:**

➢ if firms are **enough pessimistic** about the chance to meet a high quality type when they observe the unexpected signal of "education" (i.e., they are quite pessimistic about the ability of the **educational costs to discriminate among types** perhaps because universities are not able to discriminate among such types…), both workers (including the high type) will not have any incentive to get a degree

Job-Marketing Signaling: comments

- Note **how you should employ a game**:

1) you start with some assumptions (here two: a) different education costs across types and b) no value per-se of a degree for a firm);

2) you derive some equilibria according to such assumptions;

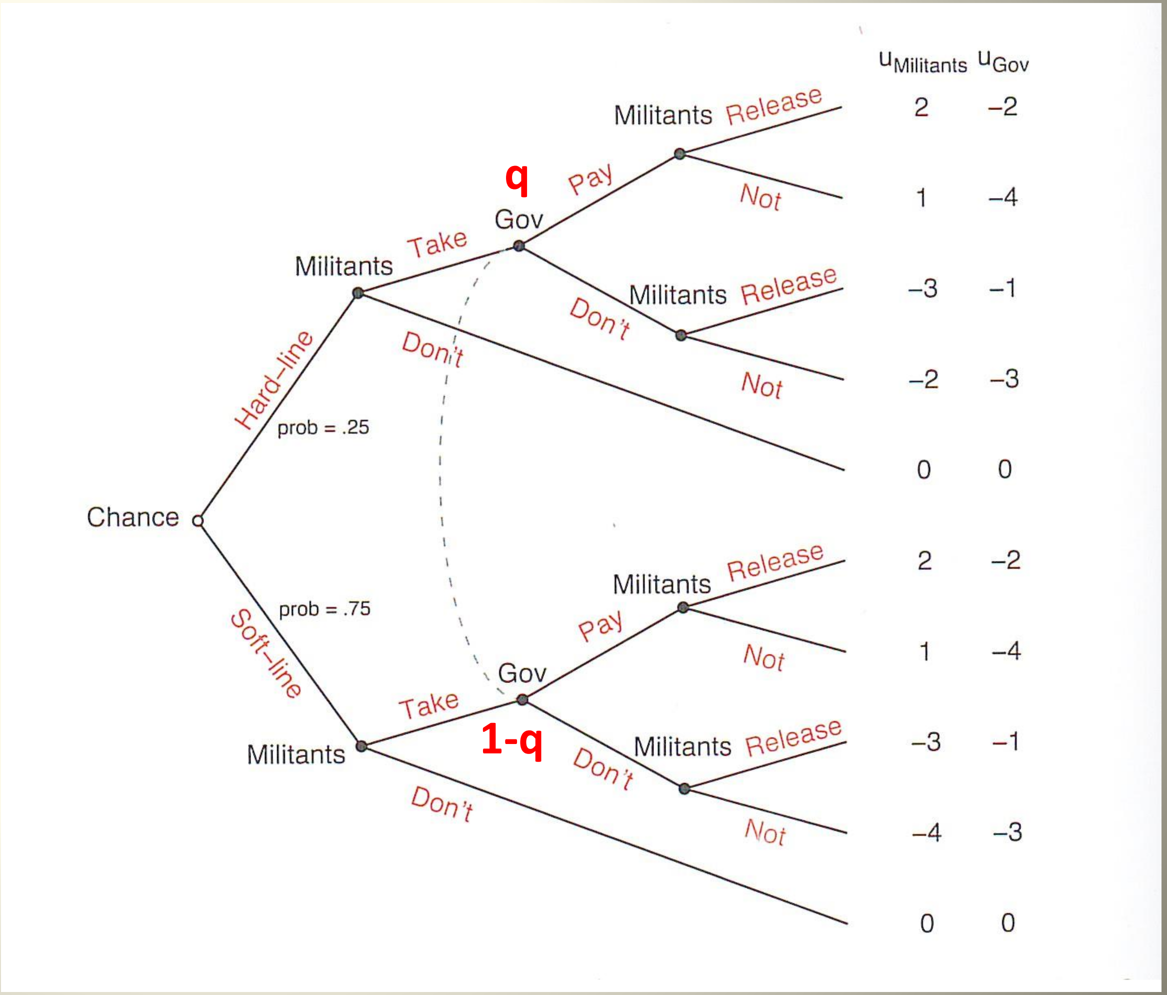
3) you derive some insights from such equilibria that could be empirically tested

Job-Marketing Signaling: comments

- Then, if the **empirical reality does not match the theoretical insights** you derive from the equilibrium (for example: it is not true that you have a lower % of university students in countries where university is free compared to countries where university is not free) you should go back to your assumptions and see if you can somehow relax them (perhaps for at least some degrees, the information you get at University is useful for firms after all)

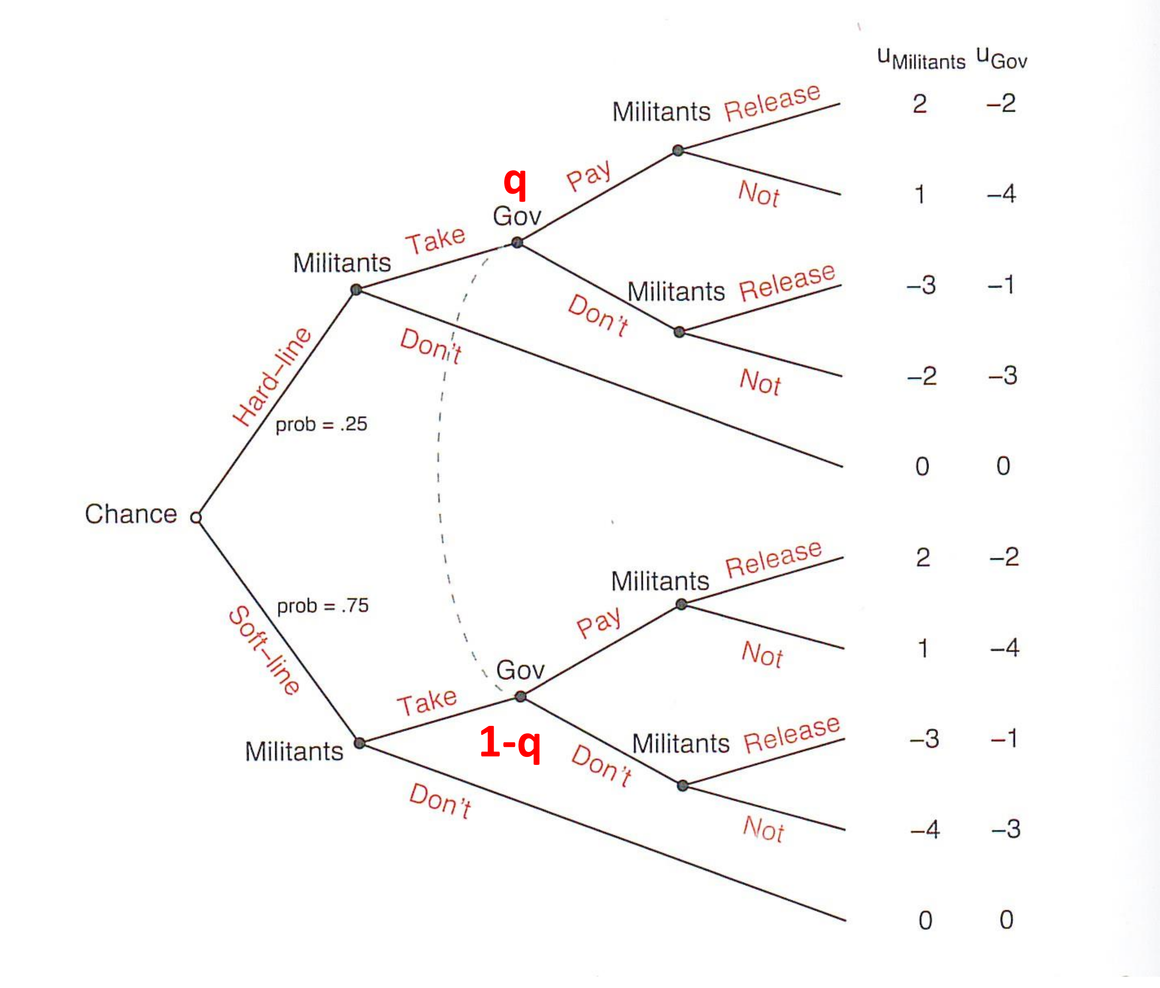
Let's go back to the ransom game

- Two players, three types
- 2 strategies available to Government (i.e., it is involved in just one information set)
- And how many strategies are available to each Militant?
- Each Militant is involved in three information sets, with 2 moves each
- Therefore: 8 strategies available to each of the 2 Militants type (2^2*2)



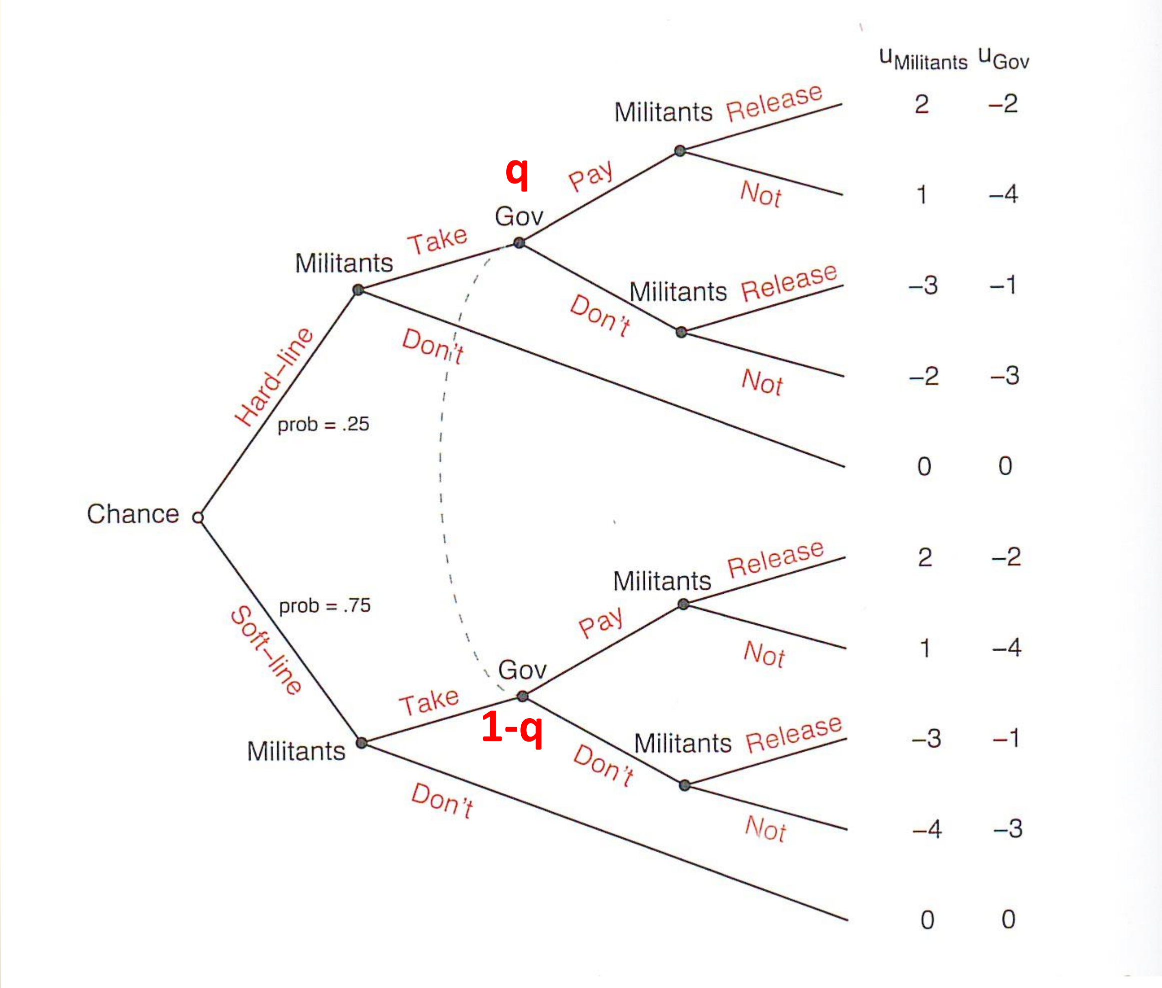
Let's go back to the ransom game

- Here **q** is the update probability that an hard-line militants is taking an hostage
- Remember that **q** can be updated only if the strategy played by at least one type of Militants reaches the information set of Government in equilibrium
- Otherwise, Bayes rule does not determine **q**
- **q** in this case represents Gov's belief about the type of Militants when the "surprise" of a kidnapping occurs (i.e., **off the equilibrium path**)



Let's go back to the ransom game

- How many proper subgames do you have?
- 2 for each type of Militants
- Therefore you can apply backward induction in part of the game!!!



Home exercise: Przeworski's model of democratic transition

- A revised version of Przeworski's model (*Democracy and the Market*, 1991)
- Przeworski's models examine the strategic interaction between liberalizers within an authoritarian government and mobilizers within civil society
- The liberalizers make the initial move , deciding between opening up the political process (open) and maintaining the status quo (**stay tough**). If a decision to stay tough is made, the outcome is a strong dictatorship (SDIC)

Przeworski's model of democratic transition

- If a decision to open up the political process is made , civil society is given an opportunity to choose between entering into a compact (**enter**) with the state or organizing politically (**organize**). If civil society enters , the outcome is broad dictatorship (BDIC)
- Given a decision to politically organize, the liberalizers must decide whether to further political **reforms** or to **repress** the organized political activity
- If the liberalizers allow further political reform, political transition to democracy (**Transition**) is the outcome

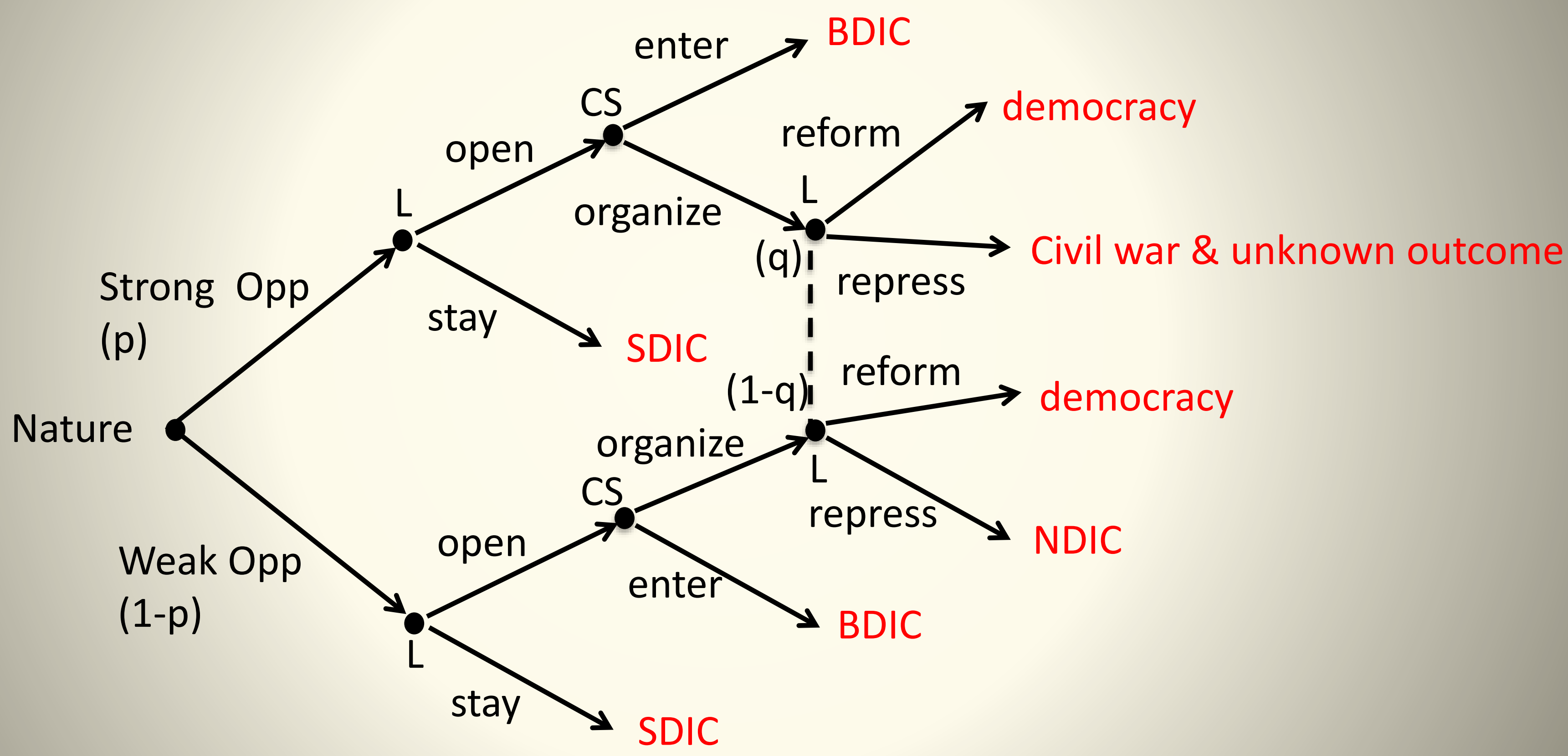
Przeworski's model of democratic transition

- However, if the liberalizers decide to repress, the outcome will depend on how strong (or weak) are the mobilizers (the opposition) within the civil society
- If the opposition is weak, then repression is successful, leading to a narrow dictatorship (NDIC)
- If the opposition is strong, then repression is unsuccessful, leading to a civil war with unknown outcomes (but surely a very costly one)
- Assuming that the civil war leads to the maintance of a «bloody» status-quo does not chance any of the results

Przeworski’s model of democratic transition

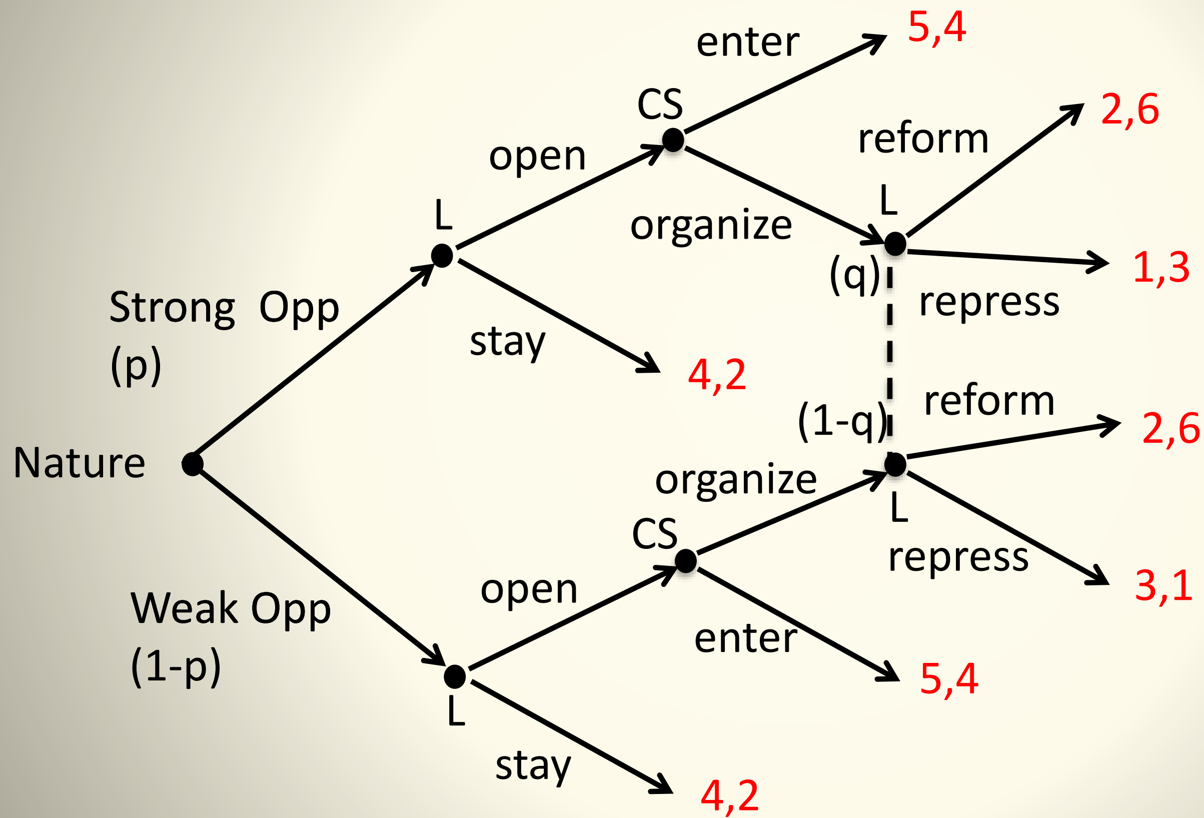
- Crucial aspect: the liberalizers do not know the type of opposition they are facing. It could be strong or weak…however, if they observe the opposition to «organize» they can update their priors

Przeworski's model of democratic transition



L = liberalizers; CS = mobilizers(opposition) within civil society

Przeworski's model of democratic transition



How many strategies are available to L?
How many strategies are available to the two types of CS?

Przeworski's model of democratic transition

- What is going to happen if we assume that the civil war leads to the maintance of a «bloody» NDIC? The game becomes trivial! Why?
- Therefore, the game does not allow in equilibrium a transition to democracy? To understand under which conditions that could happen, read Przeworski (1991)!